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# Dynamics of a Satellite Subject to Gravitational and Aerodynamic Torques. Investigation of Equilibrium Positions

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**Abstract**—The rotational dynamics of a satellite moving over a circular orbit under an effect of gravitational and aerodynamic torques is investigated. A method is proposed for determining all equilibrium positions (equilibrium orientations) of a satellite in an orbital coordinate system with given values of an aerodynamic torque vector and principal central moments of inertia; the conditions of their existence are obtained, depending on four dimensionless parameters of the problem. Bifurcation values of parameters are found for which the number of equilibrium orientations changes. The numerical analysis of the evolution of regions of existence of various numbers of equilibrium orientations in the space of dimensionless parameters is carried out. The relationship between the obtained regions of existence and the regions of existence of equilibrium orientations of an axisymmetric satellite is considered. It is shown that the number of equilibrium positions of a satellite does not exceed 24 and cannot be less than 8, in the general case.

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## 1. EQUATIONS OF MOTION

Consider the motion of a satellite, a rigid body, with respect to its center of mass in a circular orbit subject to gravitational and aerodynamic torques. To write the equations of motion, we introduce two right Cartesian coordinate systems with their origin at the center of mass $O$ of the satellite.

$OXYZ$ is the orbital coordinate system. The $OZ$ axis is directed along the radius vector connecting the centers of mass of the Earth and the satellite; the $OX$ axis is directed along the vector of linear velocity of the satellite's center of mass $O$.

$Oxyz$ is a coordinate system tied to the satellite; $Ox$, $Oy$, $Oz$ are the principal central axes of inertia of the satellite.

We define the orientation of the coordinate system $Oxyz$ with respect to the orbital coordinate system with using the Euler angles $\psi$, $\vartheta$, and $\varphi$. The direction cosines of $Ox$, $Oy$, $Oz$ axes in the orbital coordinate system are expressed in terms of Euler's angles by means of relations [1]:

$$\begin{gathered}
a_{11} = \cos(x, X) = \cos\psi\cos\varphi - \sin\psi\cos\vartheta\sin\varphi,\\
a_{12} = \cos(y, X) = -\cos\psi\sin\varphi - \sin\psi\cos\vartheta\cos\varphi,\\
a_{13} = \cos(z, X) = \sin\psi\sin\vartheta,\\
a_{21} = \cos(x, Y) = \sin\psi\cos\varphi + \cos\psi\cos\vartheta\sin\varphi,\\
a_{22} = \cos(y, Y) = -\sin\psi\sin\varphi + \cos\psi\cos\vartheta\cos\varphi,\\
a_{23} = \cos(z, Y) = -\cos\psi\sin\vartheta,\\
a_{31} = \cos(x, Z) = \sin\vartheta\sin\varphi,\\
a_{32} = \cos(y, Z) = \sin\vartheta\cos\varphi,\\
a_{33} = \cos(z, Z) = \cos\vartheta.
\end{gathered} \tag{1}$$

Then the equations of motion of the satellite with respect to its center of masses will be written as [1, 2]:

$$\begin{gathered}
A\dot{p} + (C - B)qr\\
- 3\omega_0^2(C - B)a_{32}a_{33} - \omega_0^2(H_2a_{13} - H_3a_{12}) = 0,\\
B\dot{q} + (A - C)rp\\
- 3\omega_0^2(A - C)a_{33}a_{31} - \omega_0^2(H_3a_{11} - H_1a_{13}) = 0,\\
C\dot{r} + (B - A)pq\\
- 3\omega_0^2(B - A)a_{31}a_{32} - \omega_0^2(H_1a_{12} - H_2a_{11}) = 0;
\end{gathered} \tag{2}$$

$$\begin{gathered}
p = \dot{\psi}a_{31} + \dot{\vartheta}\cos\varphi + \omega_0a_{21} = \overline{p} + \omega_0a_{21},\\
q = \dot{\psi}a_{32} - \dot{\vartheta}\sin\varphi + \omega_0a_{22} = \overline{q} + \omega_0a_{22},\\
r = \dot{\psi}a_{33} + \dot{\varphi} + \omega_0a_{23} = \overline{r} + \omega_0a_{23}.
\end{gathered} \tag{3}$$

In equations (2), (3)

$$H_1 = -Qa/\omega_0^2,\quad H_2 = -Qb/\omega_0^2,\quad H_3 = -Qc/\omega_0^2;$$

$A, B, C$ are the principal central moments of inertia of the satellite; $p$, $q$, $r$ are the projections of satellite's angular velocity on the $Ox$, $Oy$, $Oz$ axes; $\omega_0$ is the angular velocity of motion of satellite's center of masses over a circular orbit; $Q$ is the drag force acting on the satellite; $a$, $b$, $c$ are the coordinates of the center of pressure of the satellite in the coordinate system $Oxyz$. The dot indicates differentiation with respect to time $t$.

Equations (2), (3) are derived under the following assumptions [1]:

1) the effect of the atmosphere on the satellite is reduced to drag force applied at the center of pressure

and directed against the velocity of satellite's center of mass relative to the air;

2) the atmospheric effect on the translational motion of the satellite is negligible;

3) atmospheric drag by the rotation of the Earth is neglected.

Assumption 1) is rather accurately fulfilled for a shape of the satellite close to spherical.

For the equations of motion (2), (3) the generalized energy integral is valid [1]

$$\frac{1}{2}(A\bar{p}^2 + B\bar{q}^2 + C\bar{r}^2) + \frac{3}{2}\omega_0^2[(A-C)a_{31}^2 + (B-C)a_{32}^2] + \frac{1}{2}\omega_0^2[(B-A)a_{21}^2 + (B-C)a_{23}^2] - \omega_0^2(H_1a_{11} + H_2a_{12} + H_3a_{13}) = \text{const}. \quad (4)$$

## 2. EQUILIBRIUM POSITIONS OF SATELLITE

Letting $\psi = \psi_0 = \text{const}$, $\vartheta = \vartheta_0 = \text{const}$, $\varphi = \varphi_0 = \text{const}$, in (2) and (3), we obtain for $A \neq B \neq C$ the equations

$$\begin{aligned} &(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) - H_2a_{13} + H_3a_{12} = 0,\\ &(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) - H_3a_{11} + H_1a_{13} = 0,\\ &(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) - H_1a_{12} + H_2a_{11} = 0, \end{aligned} \quad (5)$$

which allow us to determine the equilibrium positions of the satellite in the orbital coordinate system. In further investigation, it is more convenient to use the equivalent system

$$\begin{aligned} &Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33} = 0,\\ &3(Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33}) + (H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0,\\ &(Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}) - (H_1a_{21} + H_2a_{22} + H_3a_{23}) = 0, \end{aligned} \quad (6)$$

which is obtained by projecting equations (5) on the axes of the orbital coordinate system. The system (6), with using the dimensionless parameters $h_i = H_i/(B-C)$ $(i = 1, 2, 3)$, $\nu = (B-A)/(B-C)$, can be presented as follows:

$$\begin{aligned} &\nu a_{21}a_{31} + a_{23}a_{33} = 0,\\ &-3(\nu a_{11}a_{31} + a_{13}a_{33}) + (h_1a_{31} + h_2a_{32} + h_3a_{33}) = 0,\\ &\nu a_{11}a_{21} + a_{13}a_{23} + (h_1a_{21} + h_2a_{22} + h_3a_{23}) = 0. \end{aligned} \quad (7)$$

With regard to (1), the system (6) or (7) can be considered the system of three equations with unknowns $\psi_0, \vartheta_0, \varphi_0$. Another way of closing equations (6) or (7) consists in adding six conditions of orthogonality of direction cosines

$$\begin{aligned} &a_{11}^2 + a_{12}^2 + a_{13}^2 = 1, \quad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,\\ &a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \quad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,\\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1, \quad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0. \end{aligned} \quad (8)$$

The system of equations (6), (8) is solved for some special cases. In papers [2, 3], for the case where the center of pressure of aerodynamic forces was located on one of principal central axes of inertia of the satellite ($h_1 \neq 0$, $h_2 = h_3 = 0$, $A \neq B \neq C$), all equilibrium positions of the satellite in the orbital coordinate system were determined analytically. For each satellite's equilibrium position, both the sufficient and necessary conditions of stability were obtained. The evolution of the domains of stability was studied, depending on two dimensionless parameters of the problem. The more complicated case, where the center of pressure of aerodynamic forces laid in one of satellite's principal central planes of inertia ($h_1 \neq 0$, $h_2 = 0$, $h_3 \neq 0$, $A \neq B \neq C$), was considered in [4]. There, a method was proposed for the numerical determination of all equilibrium positions depending on three dimensionless parameters of the problem; the sufficient conditions of their stability were obtained. And, finally, in paper [5] the equilibrium positions of the axisymmetric satellite were investigated ($h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$, $A \neq B = C$).

Further, we will investigate satellite's equilibrium positions in the general case ($h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$, $A \neq B \neq C$), using systems (6) and (8).

Equations (6) and (8) form a closed algebraic system of equations with respect to 9 unknown direction cosines that determine the equilibrium positions of a satellite. For this system of equations, the following problem is stated: for the given $A$, $B$, $C$, $H_1$, $H_2$, $H_3$, it is necessary to find all nine direction cosines, i.e., all equilibrium positions of the satellite.

As it was shown in [1, 2], the system of equations (6), (8) can be resolved with respect to $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$ for $A \neq B \neq C$ in the following manner:

$$\begin{aligned} &a_{11} = \frac{3(I_3 - A)a_{31}}{F}, \quad a_{21} = \frac{3(B-C)a_{32}a_{33}}{F},\\ &a_{12} = \frac{3(I_3 - B)a_{32}}{F}, \quad a_{22} = \frac{3(C-A)a_{33}a_{31}}{F},\\ &a_{13} = \frac{3(I_3 - C)a_{33}}{F}, \quad a_{23} = \frac{3(A-B)a_{31}a_{32}}{F}. \end{aligned} \quad (9)$$

Here $F = H_1a_{31} + H_2a_{32} + H_3a_{33}$, $I_3 = Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2$.

Substituting equations (9) into the second and third equations of (6) and adding the third equation (8), we get three equations [3, 4]

$$\begin{aligned} &9[(B-C)^2a_{32}^2a_{33}^2 + (C-A)^2a_{33}^2a_{31}^2 + (A-B)^2a_{31}^2a_{32}^2] = (H_1a_{31} + H_2a_{32} + H_3a_{33})^2,\\ &3(B-C)(C-A)(A-B)a_{31}a_{32}a_{33} - [H_1(B-C)a_{32}a_{33} + H_2(C-A)a_{33}a_{31} + H_3(A-B)\\ &\times a_{31}a_{32}](H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0,\\ &a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{aligned} \quad (10)$$

for determining the direction cosines $a_{31}, a_{32}, a_{33}$. After solving system (10), formulas (9) make it possible to determine the remained six direction cosines.

Equations (9) and (10), after the transition to dimensionless parameters, take the form

$$a_{11} = \frac{3[\nu a_{32}^2 - (1-\nu)a_{33}^2]a_{31}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \quad a_{21} = \frac{3a_{32}a_{33}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}},$$
$$a_{12} = \frac{-3(\nu a_{31}^2 + a_{33}^2)a_{32}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}},$$
$$a_{22} = \frac{-3(1-\nu)a_{33}a_{31}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \tag{11}$$
$$a_{13} = \frac{3[(1-\nu)a_{31}^2 + a_{32}^2]a_{33}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}}, \quad a_{33} = \frac{-3\nu a_{31}a_{32}}{h_1 a_{31} + h_2 a_{32} + h_3 a_{33}};$$

$$\begin{gathered} 9[a_{32}^2 a_{33}^2 + (1-\nu)^2 a_{33}^2 a_{31}^2 + \nu^2 a_{31}^2 a_{32}^2] \\ = (h_1 a_{31} + h_2 a_{32} + h_3 a_{33})^2 (a_{31}^2 + a_{32}^2 + a_{33}^2), \\ 3\nu(1-\nu)a_{31}a_{32}a_{33} - [h_1 a_{32}a_{33} - h_2(1-\nu)a_{33}a_{31} \\ - h_3 \nu a_{31}a_{32}](h_1 a_{31} + h_2 a_{32} + h_3 a_{33}) = 0, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 = 1. \end{gathered} \tag{12}$$

Note that the right-hand part of the first equation of (12) is multiplied by $a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$. Taking into account the homogeneity of the first two equations of system (12), we divide both parts of the first equation $a_{33}^4$, both parts of the second equation by $a_{33}^3$ and obtain the system of two algebraic equations with respect to variables $x = a_{31}/a_{33}$, $y = a_{32}/a_{33}$:

$$\begin{gathered} 9[y^2 + (1-\nu)^2 x^2 + \nu^2 x^2 y^2] \\ = (h_1 x + h_2 y + h_3)^2 (1 + x^2 + y^2), \\ 3\nu(1-\nu)xy - [h_1 y - h_2(1-\nu)x \\ - h_3 \nu xy](h_1 x + h_2 y + h_3) = 0. \end{gathered} \tag{13}$$

Further, substituting the expressions $a_{31} = xa_{33}$, $a_{32} = ya_{33}$ into the last equation of system (12), we obtain the expression

$$a_{33}^2 = \left(1 + x^2 + y^2\right)^{-1}. \tag{14}$$

The system of equations (13) can be represented in the form

$$\begin{gathered} a_0 y^2 + a_1 y + a_2 = 0, \\ b_0 y^4 + b_1 y^3 + b_2 y^2 + b_3 y + b_4 = 0, \end{gathered} \tag{15}$$

where

$$\begin{gathered} a_0 = h_2(\nu h_3 x - h_1), \quad a_1 = \nu h_1 h_3 x^2 - h_1 h_3 \\ + [3\nu(1-\nu) - h_1^2 + (1-\nu)h_2^2 + \nu h_3^2]x, \quad a_2 = (1-\nu)h_2 \\ \times (h_1 x + h_3)x, \quad b_0 = h_2^2, \quad b_1 = 2h_2(h_1 x + h_3), \\ b_2 = (h_2^2 + h_3^2 - 9) + 2h_1 h_3 x + (h_1^2 + h_2^2 - 9\nu^2)x^2, \\ b_3 = 2h_2(h_1 x + h_3)(1 + x^2), \\ b_4 = (h_1 x + h_3)^2(1 + x^2) - 9(1-\nu)^2 x^2. \end{gathered} \tag{16}$$

The resultant $R(x)$ of equations (15) has the following form:

$$R(x) = \begin{bmatrix} a_0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}.$$

The equation $R(x) = 0$, using the symbolic functions of the computer algebra system Mathematica, can be presented in the form

$$\begin{gathered} p_0 x^{12} + p_1 x^{11} + p_2 x^{10} + p_3 x^9 + p_4 x^8 + p_5 x^7 + p_6 x^6 \\ + p_7 x^5 + p_8 x^4 + p_9 x^3 + p_{10} x^2 + p_{11} x + p_{12} = 0, \end{gathered} \tag{17}$$

where

$$\begin{gathered} p_0 = h_1^4 h_3^4 \nu^6, \\ p_1 = -2h_1^3 h_3^3 \nu^5 \left[2h_1^2 + (1-\nu)h_2^2 - 2\nu h_3^2 - 3\nu(1-\nu)\right], \\ p_2 = h_1^2 h_3^2 \nu^4 [6h_1^4 + h_1^2(h_2^2(\nu^2 - 8\nu + 7) \\ + h_3^2(\nu^2 - 16\nu + 1) + 17\nu^2 - 16\nu - 1) + h_2^4(1-\nu)^2 \\ + h_2^2(1-\nu) + 3\nu^2(2h_3^4 + 3h_3^2(1-\nu^2) + 3(1-\nu)^2)], \\ \dots \\ p_{10} = h_1^2 h_3^2 [6h_1^4 + h_1^2(h_2^2(\nu^2 - 8\nu + 7) \\ + h_3^2(\nu^2 - 16\nu + 1) - 9(1-\nu^2) + h_2^4(1-\nu)^2 \\ + h_2^2(1-\nu)(h_3^2(1-7\nu) + \nu^3 - \nu^2 + 9\nu - 9) \\ + \nu^2(6h_3^4 - h_3^2(\nu^2 + 16\nu - 17) + 9(1-\nu)^2)], \\ p_{11} = 2h_1^3 h_3^3 \left[2h_1^2 + (1-\nu)h_2^2 - 2\nu h_3^2 - 3\nu(1-\nu)\right], \\ p_{12} = h_1^4 h_3^4. \end{gathered}$$

Coefficients $p_i$ $(i = 3, \dots, 9)$ represent very cumbersome expressions (these coefficients are presented completely in [6]).

The number of real roots of the obtained algebraic equation (17) is even and does not exceed 12. Substituting the value of the real root $x_1$ of equation (17) into the equations of system (15), we find the coinciding root $y_1$ of these equations. For each solution $(x_1, y_1)$, we can determine two values of $a_{33}$ from equation (14), and then find the quantities $a_{31} = x_1 a_{33}$ and $a_{32} = y_1 a_{33}$ corresponding to these values of $a_{33}$. Thus, each real root of the algebraic equation (17) is associated with two sets of values $a_{31}, a_{32}, a_{33}$, which, in virtue of (11), uniquely determine the remaining direction cosines $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}$. It follows from the above considerations that the satellite in a circular orbit, subject to gravitational and aerodynamic torques, can have not more than 24 equilibrium positions in the general case ($h_1 \neq 0, h_2 \neq 0, h_3 \neq 0, A \neq B \neq C$).

## 3. STUDY OF EQUILIBRIUM POSITIONS OF THE SATELLITE

Equations (15) and (17), together with systems (11) and (12), allow us to determine all equilibrium positions of the satellite subjected to gravitational and aerodynamic torques for the given values of problem's parameters.

To study the satellite's equilibrium positions, we state the problem of finding, in the parametric space, regions with an identical number of real roots of equation (17). The separation of the parametric space into regions with an identical number of real roots is determined by the discriminant hypersurface, which is specified by the discriminant of polynomial (17). The system of algebraic equations that defines the set of singular points of the discriminant hypersurface cannot be investigated in the symbolic form because of the cumbersome character of the expressions for the coefficients of polynomial (17).

The dependence of the number of real solutions of equation (17) on the values of parameters was studied numerically using the factorization package of the system Mathematica 8.0, which allows us to calculate the roots of algebraic equations to a specified accuracy.

Without losing generality, numerical investigations can be performed under the condition $B > A > C$, then $0 < \nu < 1$. Projections of the aerodynamic torque vectors $h_1, h_2, h_3$ can assume any nonzero values.

Coefficients of equation (17) depend on 4 dimensionless parameters $\nu, h_1, h_2, h_3$, and the equations of the original system (6) include 6 parameters: $H_1, H_2, H_3, A, B, C$. In the numerical investigation of the problem, the decrease of the number of parameters is an essential factor.

As it was shown in [5], for the extreme cases $\nu = 0$ and $\nu = 1$ (cases of the axisymmetric satellite) the boundaries between the regions with a constant number of equilibrium positions are determined analytically.

For the axisymmetric case $\nu = 0$ ($A = B$), the system of equations (7) is simplified, and, as a result, one can obtain equations of two circles in the plane $(h_1, h_2)$, which determine the boundaries of regions with a constant number of equilibrium positions of the satellite:

$$h_1^2 + h_2^2 = (3^{2/3} - h_3^{2/3})^3, \quad h_1^2 + h_2^2 = (1 - h_3^{2/3})^3. \qquad (18)$$

For the axisymmetric case $\nu = 1$ ($A = C$), the system of equations (7) is also simplified, and, as a result, one can obtain the equations of two asteroids in the plane $(h_1, h_2)$, which determine the boundaries of regions with a constant number of equilibrium positions of the satellite:

$$h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 3^{2/3}, \quad h_2^{2/3} + (h_1^2 + h_3^2)^{1/3} = 1. \qquad (19)$$

Consider now in more detail the properties of the algebraic equation (17). From the form of the coefficients of equation (17), presented in [6], it follows that the number of real roots does not depend on the signs of parameters $h_1, h_2, h_3$. Indeed, in the expressions for the coefficients of equation (17) for even powers of $x$ $p_{2k}$ ($k = 1, 2, 3, 4, 5, 6$), parameters $h_1, h_2, h_3$ appear at even powers only, and the coefficients at odd powers of $x$ $p_{2k+1}$ ($k = 1, 2, 3, 4, 5$), represent the product of factors $p_{2k+1} = h_1 h_3 P_{2k+1}$, where the multiplier $P_{2k+1}$ depends only on the even powers of parameters $h_1$, $h_2$, $h_3$. Therefore, in changing the signs of parameters $h_1, h_2, h_3$, only the sign of the product $h_1 h_3$ can change, and, thus, the sign of the real roots of equation (17) can change. In this case, the magnitudes of real roots and their number remain unchanged.

The numerical analysis of the number of real roots of equation (17) was performed with positive values of $h_1, h_2, h_3$ and under the condition $0 < \nu < 1$. Calculations were carried out at the nodes of the uniform grid on the plane $(h_1, h_2)$ with fixed values of $\nu$ and $h_3$. The boundary points at which the number of real roots changed were determined numerically. The two-dimensional cross section of the discriminant hypersurface, which was implicitly specified by the algebraic equation of two parameters $g(h_1, h_2) = 0$, was calculated. It was found experimentally that for obtaining smooth boundary curves, calculations should be performed with a grid step of 0.0001. Calculations with such accuracy become very labor consuming. Indeed, for the region of size $3 \times 3$ on the $(h_1, h_2)$ plane the calculation of roots should be performed at $10^9$ nodes. For this reason the calculations were performed in two stages. At the first stage, the number of real roots of equation (17) was determined at $10^7$ nodes with a step of 0.001. At the second stage, the number of real roots was determined in the neighborhood of an approximately calculated boundary between regions with a constant number of real roots at grid nodes with a step of 0.0001.

Below, for fixed values of $h_2$, the value of boundary points $h_1$ between two regions with various constant number of real roots was determined with specified

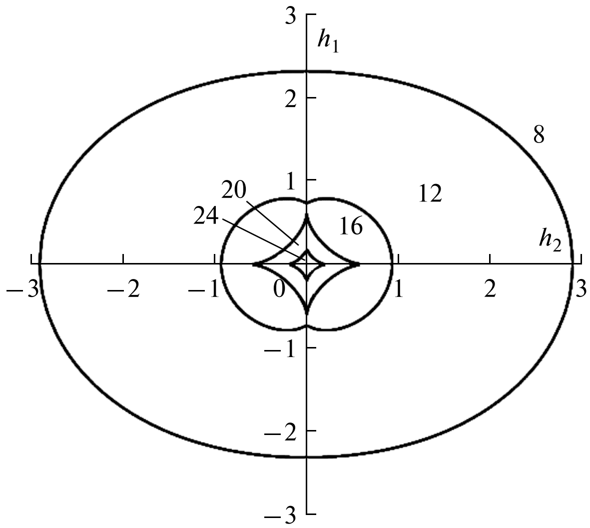


**Fig. 1.** Regions of existence of equilibrium orientations ($\nu = 0.2$, $h_3 = 0.01$).

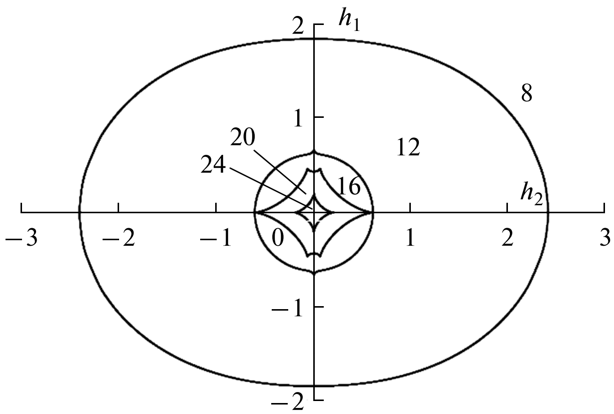


**Fig. 2.** Regions of existence of equilibrium orientations ($\nu = 0.2$, $h_3 = 0.15$).

accuracy by the bisection method, which was implemented in the programming language of the Mathematica system in a package form. Numerical methods of solving the equations that were implemented in the Mathematica system allow the calculation of the roots of the algebraic equation with very small values of coefficients.

Figures 1–10 present the results of calculations of the evolution of boundaries between the regions with an equal number of real roots on the $(h_1, h_2)$ plane for the values $\nu = 0.01$ (in the neighborhood of the axisymmetric case for $\nu = 0$), $\nu = 0.2$, $\nu = 0.5$, and $\nu = 0.8$.

The analysis of the numerical results for the specified parameters $\nu = 0.01$ indicates that, as parameter $h_3$ increases, the size of the regions with a constant number of real roots decreases. The points in the parametric space beginning at which the regions with a particular number of real roots disappear will be called bifurcation points. The results of calculations of bifurcation values of parameters are presented in the table.

The calculations in Figs. 1–10 were performed for the bifurcation values of parameter $h_3$, indicated in the table, and for $h_3$ values corresponding to the mean value of the distance between two adjacent bifurcation points.

It follows from the table that the bifurcation values of parameter $h_3$ at which the regions of existence of 24 equilibrium solutions (12 real roots) disappear satisfy the relation $h_3 = 1 - \nu$.

The bifurcation values of parameter $h_3$ at which the regions of existence of 20 equilibrium solutions (10 real roots) disappear are equal to 1 with increasing $\nu$ up to the value $\nu = 0.6$, after which they decrease in accordance with the relation $h_3 = 3(1 - \nu)$.

For the regions where there exist 16 equilibrium solutions (8 real roots), the bifurcation values of parameter $h_3$ decrease in accordance with the relation

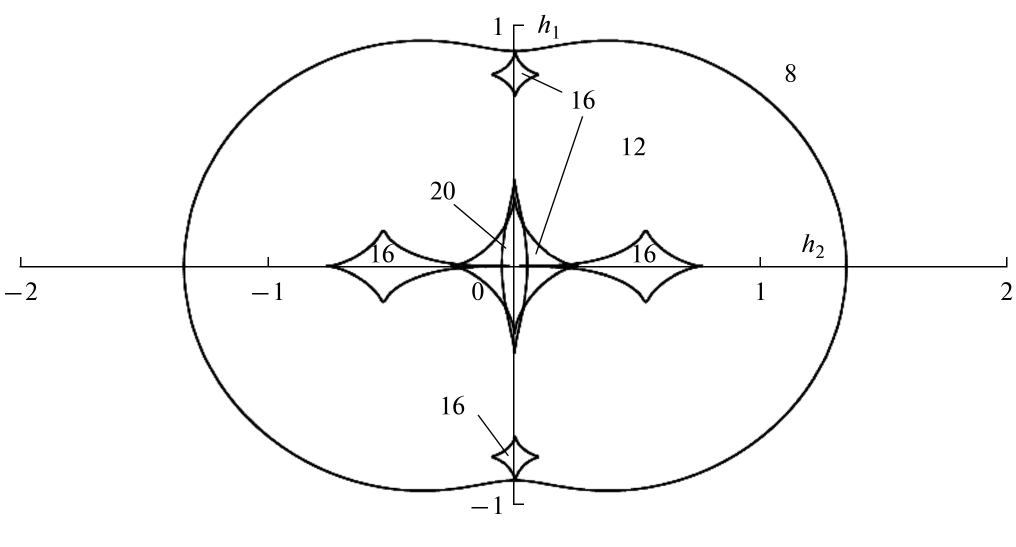


**Fig. 3.** Regions of existence of equilibrium orientations ($\nu = 0.2$, $h_3 = 0.8$).

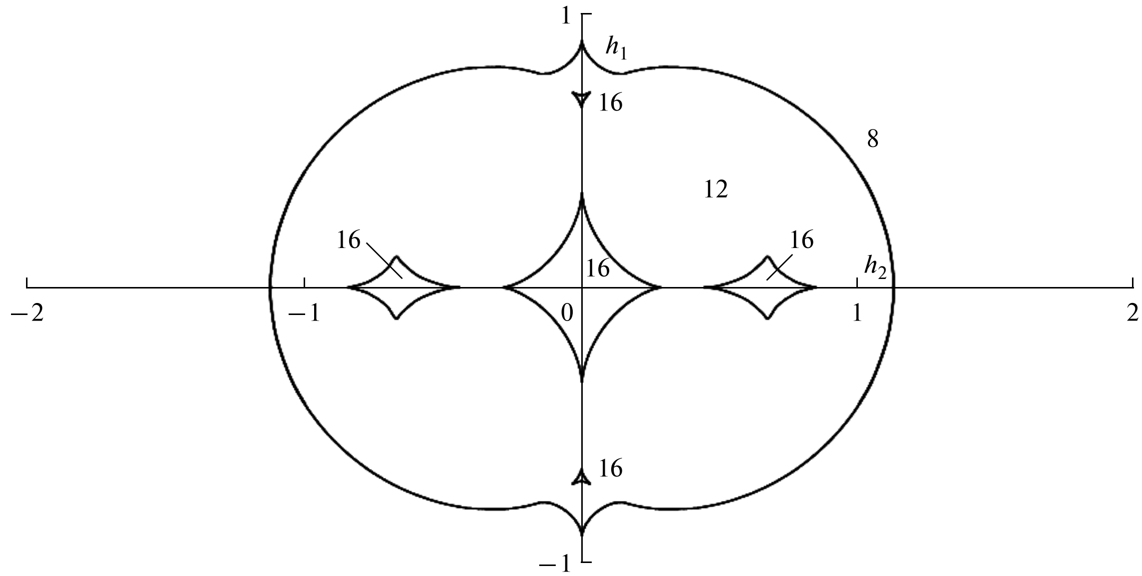


**Fig. 4.** Regions of existence of equilibrium orientations ($\nu = 0.2$, $h_3 = 1.0$).

$h_3 = 3(1 - \nu)$ with increasing $\nu$ up to the value $\nu = 0.6$, after which they remain at 1.

The regions with a number of equilibrium positions equal to 12 decrease with increasing the value of parameter $h_3$. The central part of these regions disappears at $h_3 = 3$. For the values $h_3 \geq 3$ there exist small regions whose number of equilibrium positions is also 12; these regions are located near the $Oh_2$ axis with characteristic dimensions along the $Oh_1$ and $Oh_2$ axes not exceeding the value of $10^{-1}$. As the value of $h_3$ increases, these regions decrease and shift to the right along the positive part of axis $Oh_2$ and to the left along the negative part of axis $Oh_2$.

Consider the example in the neighborhood of the axisymmetric case $\nu = 0$) for the value $\nu = 0.01$. In this case, calculations were carried out for the value of parameter $h_3 = 0.01$ (in the neighborhood of zero value). The curves in this case are very similar to corresponding curves for the axisymmetric case for the axisymmetric case $\nu = 0$, which are determined by equations (18). As shown in paper [5], in the case of axisymmetric satellite the number of equilibrium positions may only be 16, 12, and 8, and, accordingly,

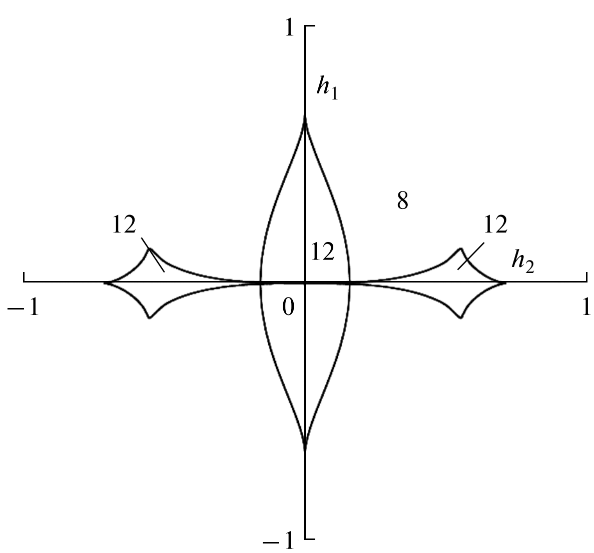


**Fig. 5.** Regions of existence of equilibrium orientations ($\nu = 0.2$, $h_3 = 2.4$).

Bifurcation values of $\nu$, $h_3$

| $\nu$ | $h_3$ (24/20) | $h_3$ (20/16) | $h_3$ (16/12) | $h_3$(12/8) |
|---|---|---|---|---|
| 0.01 | 0.99 | 1.0 | 2.97 | 3.0 |
| 0.1 | 0.90 | 1.0 | 2.7 | 3.0 |
| 0.2 | 0.80 | 1.0 | 2.4 | 3.0 |
| 0.3 | 0.70 | 1.0 | 2.1 | 3.0 |
| 0.4 | 0.60 | 1.0 | 1.8 | 3.0 |
| 0.5 | 0.50 | 1.0 | 1.5 | 3.0 |
| 0.6 | 0.40 | 1.0 | 1.2 | 3.0 |
| 0.7 | 0.30 | 0.9 | 1.0 | 3.0 |
| 0.8 | 0.20 | 0.6 | 1.0 | 3.0 |
| 0.9 | 0.10 | 0.3 | 1.0 | 3.0 |
| 0.99 | 0.01 | 0.03 | 1.0 | 3.0 |

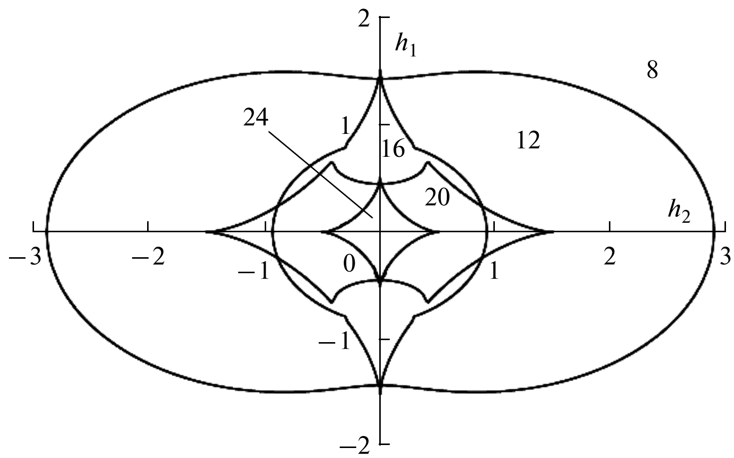


**Fig. 6.** Regions of existence of equilibrium orientations ($\nu = 0.5$, $h_3 = 0.01$).

there exist only 3 regions in the parametric space with an equal number of equilibrium positions. In this case we have only two bifurcation values of parameter $h_3$: $h_3 = 1$ and $h_3 = 3$.

For values of the inertial parameter $\nu = 0.99$, which is close to the axisymmetric case $\nu = 1$, the boundaries of regions were calculated for the values $h_3 = 0.01$ (the bifurcation point where the region with 24 equilibrium positions disappears). The boundary curves for the values of inertial parameter tending to 1, approach the corresponding analytical curves for the axisymmetric case $\nu = 1$ which are determined by equations (19). In this case we have also two bifurcation values for parameter $h_3$: $h_3 = 1$ and $h_3 = 3$.

Within the interval of inertial parameter values $0.1 \le \nu \le 0.9$ the evolution of regions with a constant number of equilibrium positions of 24, 20, 16, 12, and 8 has been investigated numerically (Figs. 1–10). Let us consider in more detail the character of the change of the regions with the number of equilibrium positions equal to 24, 20, 16, 12, and 8, for the example, when $\nu = 0.2$ (Figs. 1–5).

The analysis of the numerical results shows that for $\nu = 0.2$, regions with a number of equilibrium positions equal to 24, 20, 16, 12, and 8 exist in the $(h_1, h_2)$ plane for $h_3 < 0.8$ (Figs. 1, 2). It is seen from Fig. 2 that, as the value of $h_3$ increases, the size of regions with a number of equilibrium positions of 24, 20, 16, and 12 becomes smaller, than corresponding regions in Fig. 1. For the bifurcation value $h_3 = 0.8$, the region with the number of equilibrium positions equal to 24 disappears (Fig. 3), and within the interval of values $0.8 < h_3 < 1.0$ there exist only four types of regions with the number of equilibrium positions equal to 20, 16, 12, and 8. For the bifurcation value $h_3 = 1.0$ the region with a number of equilibrium positions of 20 disappears (Fig. 4). Within the interval of values $1.0 \le h_3 < 2.4$ there exist only three types of regions

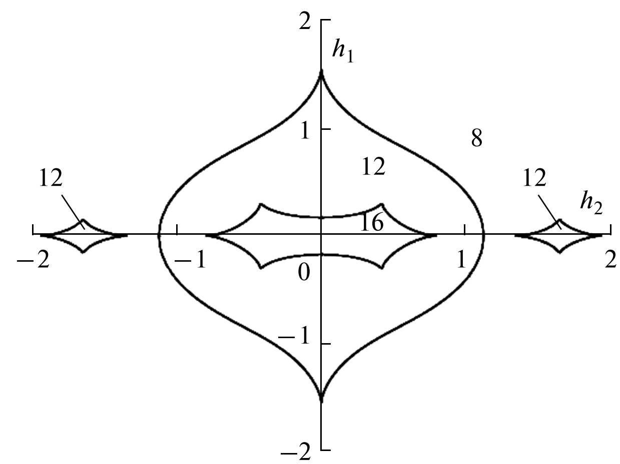


**Fig. 7.** Regions of existence of equilibrium orientations ($\nu = 0.5$, $h_3 = 1.0$).

with a number of equilibrium positions equal to 16, 12, and 8 (Fig. 4).

For the bifurcation value $h_3 = 2.4$, the region with the number of equilibrium positions equal to 16 disappears (Fig. 5). Within the interval $2.4 \le h_3 < 3$, there remain only two types of regions with a number of equilibrium positions equal to 12 and 8. For $h_3 = 3.0$ regions with the number of equilibrium positions equal to 12 disappear in the neighborhood of the coordinate origin, and, with further increases of parameter $h_3$ values near the $Oh_2$ axis, there appear small regions with the number of equilibrium positions equal to 12.

Figures 6–10 present the evolution of regions with the constant number of equilibrium positions for the values of inertial parameters $\nu = 0.5$ and $\nu = 0.8$.

When the values of the aerodynamic torque parameter $h_3$ are greater than 3, for any values of parameters $h_1$ and $h_2$ there exist 8 equilibrium posi-

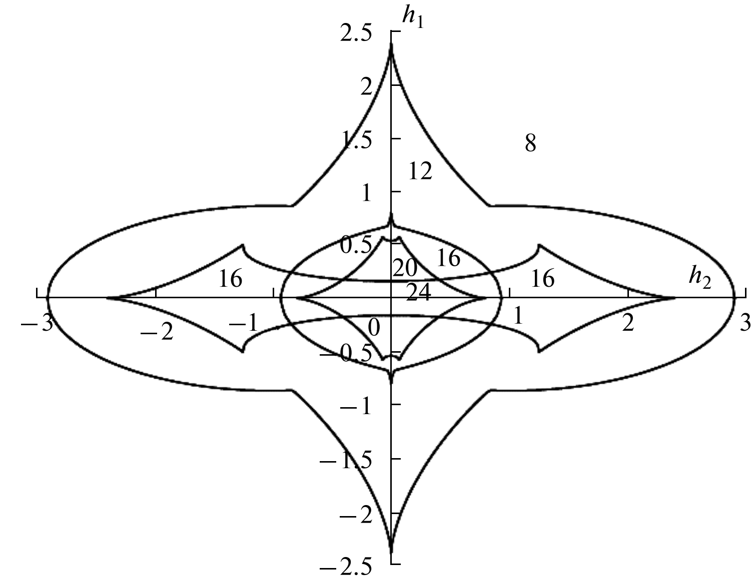


**Fig. 8.** Regions of existence of equilibrium orientations ($\nu = 0.8$, $h_3 = 0.01$).

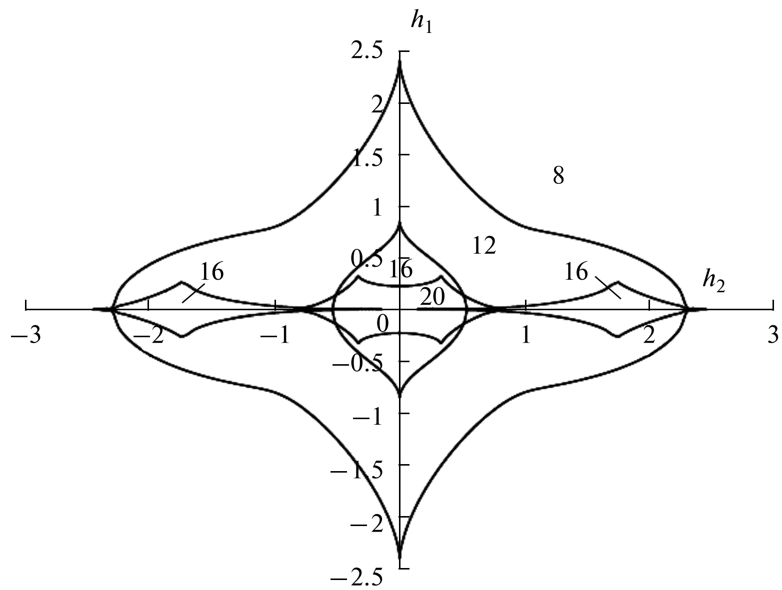


**Fig. 9.** Regions of existence of equilibrium orientations ($\nu = 0.8$, $h_3 = 0.2$).

tions of the satellite, which correspond to 4 real roots of equation (17).

## CONCLUSIONS

In the present work, we have studied the rotational motion of the satellite with respect to its center of mass in a circular orbit subject to gravitational and aerodynamic torques. Main attention was given to determination of the equilibrium positions of the satellite in the orbital coordinate system. A symbolic–numerical method of determining all equilibrium positions of the satellite in the orbital coordinate system was proposed for the specified values of the aerodynamic torque vector and principal central moments of inertia in the general case, when $A \neq B \neq C$ and $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$.

A detailed numerical analysis of the evolution of regions of existence of various numbers of equilibrium positions in the plane of two parameters $(h_1, h_2)$ was per-

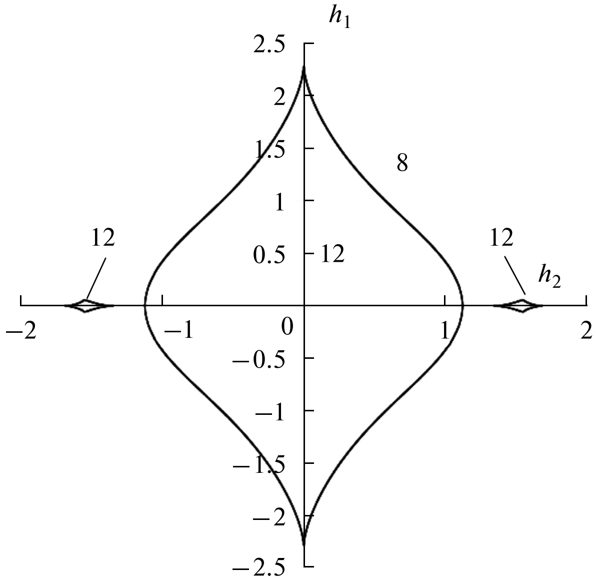


**Fig.10.** Regions of existence of equilibrium orientations ($\nu = 0.8$, $h_3 = 1.0$).

formed for various values of parameters $\nu$ and $h_3$. It was shown that the number of the satellite's equilibrium positions in a circular orbit does not exceed 24 and cannot be less than 8, in the general case. The obtained results can be used at the stage of preliminary design of the aerodynamic attitude control system of the satellite.

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