Application of Computer Algebra to Investigation of Equilibrium Orientations of Satellite Subject to Gravitational and Constant Torques

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Abstract. *The equilibrium orientations of a satellite in a circular orbit under the influence of gravitational and constant torques are investigated. The stationary motions of a satellite are governed by a set of nonlinear algebraic equations. A computer algebra method based on algorithm for the construction Groebner basis is proposed for determining the equilibrium orientations of a satellite with a given constant torque vector and given principal central moments of inertia. It is shown that 24 isolated equilibrium orientations exist when the module of constant vector is sufficiently small. The equilibrium orientations are determined by algebraic equation of the sixth degree and are analyzed numerically.*

1 Introduction

Determination of equilibrium orientations of a satellite under the action of external torques is one of the basic problems of astrodynamics. Such solutions are used in the design of rather simple, cheap and long-living passive attitude control systems of satellites. The problem to be analyzed in the present report is related to behavior of a satellite acted upon by the gravity gradient and constant torques. The constant torque may be produced actively or caused, for example, by gas or fuel escape from the satellite. In the absence of constant torque there are 24 equilibrium orientations of satellite [1], [2] and [3]. The action of some constant torque change this orientations and can destroy some or even all these equilibria. In [4] the existence of equilibria for a satellite under the action of the gravity gradient and constant disturbing torques in some particular cases was indicated. For general values of constant torque this problem was studied in [5], using aircraft angles approach for determining of satellite equilibrium orientations. It was shown that for small constant torque there exist 24 equlibria, and the number of equlibria decreases with the increase of constant torque. Using the above approach, the classification of different distributions of the number of equilibria as the function of parameters of the problem, namely, the components of the constant torque, the inertial parameters of the satellite and the angular velocity of its orbital motion was done in [6]. A different method to solve this problem was suggested in [7].

The problem of determining the classes of equilibrium orientations for general values of constant torque which are determined by real roots of the system of non-linear algebraic equations is considered in this paper. The investigation a number of equilibria developing both the Computer Algebra Groebner basis method and the numerical analysis of these algebraic equations has completed. Evolution of domains with fixed number of equilibria has investigated numerically in dependence of three dimensionless system parameters and bifurcation values of the system parameters corresponding to the qualitative change of these domains have been determined.

2 Equations of motion

Consider the motion of a satellite subjected to gravitational and constant torques in a circular orbit. We assume that 1) the gravity field of the Earth is central and Newtonian, 2) the satellite is a triaxial rigid body, 3) the satellite is subjected to gravity gradient torque and a torque that is fixed with respect to the body of satellite, so the components of this torque in the body fixed frame are constant. We introduce two right handed Cartesian coordinate systems with origin in the satellites center of mass O. $OXYZ$ is the orbital reference frame. The axis OZ is directed along the radius vector from the Earth center of mass to the satellites one, the axis OX is in the direction of satellites orbital motion. $Oxyz$ is the satellite body reference frame; Ox, Oy, Oz are the principal central axes of inertia of the satellite. The orientation of the satellite body reference frame $Oxyz$ with respect to the orbital reference frame is determined by the angles of pitch (α), yaw (β) and roll (γ), and the direction cosines of the axis Ox, Oy, Oz in the orbital reference frame can be written as [1]

$$\begin{aligned}
a_{11} &= \cos(x, X) = \cos\alpha\cos\beta, \\
a_{12} &= \cos(y, X) = \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma, \\
a_{13} &= \cos(z, X) = \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma, \\
a_{21} &= \cos(x, Y) = \sin\beta, \\
a_{22} &= \cos(y, Y) = \cos\beta\cos\gamma, \\
a_{23} &= \cos(z, Y) = -\cos\beta\sin\gamma, \\
a_{31} &= \cos(x, Z) = -\sin\alpha\cos\beta, \\
a_{32} &= \cos(y, Z) = \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma, \\
a_{33} &= \cos(z, Z) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\sin\gamma.
\end{aligned} \tag{1}$$

Then equations of the satellites attitude motion take the form [5]:

$$\begin{aligned}
A\dot{p} + (C - B)qr - 3\omega_0^2(C - B)a_{32}a_{33} - \tilde{a} &= 0, \\
B\dot{q} + (A - C)rp - 3\omega_0^2(A - C)a_{31}a_{33} - \tilde{b} &= 0, \\
C\dot{r} + (B - A)pq - 3\omega_0^2(B - A)a_{31}a_{32} - \tilde{c} &= 0;
\end{aligned} \tag{2}$$

$$
\begin{aligned}
p &= (\dot{\alpha} + \omega_0)a_{21} + \dot{\gamma}, \\
q &= (\dot{\alpha} + \omega_0)a_{22} + \dot{\beta}\sin\gamma, \\
r &= (\dot{\alpha} + \omega_0)a_{23} + \dot{\beta}\cos\gamma.
\end{aligned} \tag{3}
$$

Here A, B, C are the principal central moments of inertia of the satellite; p, q, r are the projections of the angular velocity of the satellite in the axes Ox, Oy, Oz; ω_0 is the angular velocity of the center of mass of the satellite, while $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are the components of the constant torque in the same frame. The dot designates differentiation with respect to time t.

3 Equilibrium orientations of a satellite

Putting in (2) and (3) $\alpha = \alpha_0$, $\beta = \beta_0$, $\gamma = \gamma_0$, (α_0, β_0, γ_0 are constants) we get at $A \neq B \neq C$ the equations

$$
\begin{aligned}
(a_{22}a_{23} - 3a_{32}a_{33}) &= a, \\
(a_{21}a_{23} - 3a_{31}a_{33}) &= b, \\
(a_{21}a_{22} - 3a_{31}a_{32}) &= c.
\end{aligned} \tag{4}
$$

Here

$$
a = \frac{\tilde{a}}{\omega_0^2(c - B)}, \quad b = \frac{\tilde{b}}{\omega_0^2(A - C)}, \quad c = \frac{\tilde{c}}{\omega_0^2(B - A)}
$$

are the constants that characterize the dimensionless components of the constant torque.

The system (4) with the following three orthogonality conditions for the direction cosines

$$
\begin{aligned}
a_{21}^2 + a_{22}^2 + a_{23}^2 &= 1, \\
a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1, \\
a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0
\end{aligned} \tag{5}
$$

can be considered as a system of six algebraic equations for six unknown direction cosines, which allow us to determine the satellite equilibria in the orbital reference frame. After a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} are found, the direction cosines a_{11}, a_{12} and a_{13} can be determined from the conditions of orthogonality.

We state the following problem for the system of equations (4), (5): determine all equilibrium orientations of a satellite when A, B, C, a, b, c are given. It is possible to see at once that for any given attitude of the satellite there will always exist a, b and c such that this attitude is an equilibrium orientation. Analytical solution of the problem of defining all equilibrium orientations of a satellite in general case ($a \neq 0$, $b \neq 0$, $c \neq 0$,) is impossible. In a previous paper [8] the system of equations (4), (5) has been solved for a_{2i} and a_{3i} ($i = 1, 2, 3$) and then reduced to three equations extremely complex form in a_{11}, a_{12} and a_{13} which were found to be unsolvable. In [5], using the direction cosines in terms of orientation angles (1) and concept of resultant, is shown that the system of equations (4), (5) can be reduced to a single algebraic equation of the sixth degree with real coefficients, which represent polynomials depending on three dimensionless parameters of the system. To every real root of this algebraic equation corresponds four equilibrium orientations of a satellite. Since the number of real roots of the algebraic equation of the sixth degree does not exceed 6 therefore the satellite subjected to gravitational and constant torques can have no more than 24 equilibrium orientations in a circular orbit.

In the case $a = b = c = 0$ it has been proved that the system (4), (5) has 24 solutions describing the equilibrium orientations of a satellite-rigid body. Further we consider a Computer Algebra approach to define equilibrium orientations of a satellite. A solution of the system (4), (5) can be obtained using an algorithm for the construction of Groebner bases [9]. The method of Groebner bases is used to solve systems of nonlinear algebraic equations. It comprises an algorithmic procedure for reducing the problem involving polynomials of several variables to investigation of a polynomial of one variable. Using a computer algebra system Maple we calculate the Groebner basis of the system (4), (5) under ordering on the total power of variables. Here we write out the polynomials in the Groebner basis that depend only on the variables a_{22}, a_{31}, a_{32} and a_{33}:

$$\begin{aligned}
&ba_{22}^2 - 3ba_{32} + 12a_{31}a_{32} - ac + 3b = 0,\\
&144a_{32}^2a_{33}^2 + (84a - 24bc)a_{32}a_{33} + 12a^2(a_{32}^2 + a_{33}^2)\\
&+ a^2c^2 + a^2b^2 + b^2c^2 - 7abc = 0,\\
&12ca_{33}^3 + 12ba_{32}a_{33}^2 - b(c^2 + 12)a_{32} - c(b^2 + 12)a_{33}\\
&+ 12ca_{33}a_{32}^2 + 12ba_{32}^3 + (ac^2 + ab^2 - 7bc)a_{31} = 0,\\
&12aa_{31}a_{33}^2 + 12ba_{32}a_{33}^2 + a^2ba_{32} + ab^2a_{31} - (cb^2 + ca^2 - 7ab)a_{33} = 0.
\end{aligned} \tag{6}$$

The equation in the Groebner basis that depends only from one variable a_{33}^2 has the form

$$p_0x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6 = 0. \tag{7}$$

Here the coefficients p_i $(i = 0, 1, 2, 3, 4, 5, 6)$ are

$$\begin{aligned}
&p_0 = 4096,\\
&p_1 = -8192,\\
&p_2 = 256(16 - 6u^2 + 20uc + 17v),\\
&p_3 = -128(u^2v + 8c^2u^2 - 8u^2 + 40cu + 5cuv + 34v - c^2v^2),\\
&p_4 = 16(9u^4 + 257u^2 - 43u^2v + 16v^2 + 210cuv - 20cuv^2 + 20cu^3 + 17v^2 + 64c^2u^2),\\
&p_5 = 8(4u^2v^2 - 34u^2v - 38u^4 - 3u^4v - 130cu^3 + 5cu^3v - 20cuv^2 - 4c^2v^3\\
&- 68c^2u^2v + 3c^2u^2v^2 - 10c^3uv^2),\\
&p_6 = (4u^2 + u^2v + c^2v^2 + 5cuv)^2, \quad v = a^2 + b^2, \quad u = ab.
\end{aligned}$$

Equation (7) together with (6) and (4) can be used to determine all the equilibrium orientations of a satellite under the influence of gravitational and constant torques. Using the equations from the Groebner basis (6), we can show that every real root of equation (7) corresponds to four equilibrium solutions of system (4), (5). Since the number of real roots of equation (7) does not exceed 6, the number of equilibrium orientations can be no more than 24. Analysis of special cases when only one parameter has a nonzero value (the constant torque vector coincides with the principal axes of inertia of the satellite) say $a \neq 0$, $b = c = 0$, was studied in [5] another special cases, one of them for example $b = c \neq 0$, was presented in [6]. Evolution of domains with fixed number of equilibria was investigated numerically from (7) in dependence of tree system parameters a, b, c. The regions of the (a, b, c) space where equilibria exist are limited by the inequalities [5]

$$a^2 + b^2 + c^2 \leq 5, \quad a^2 + b^2 \leq 4, \quad c^2 + b^2 \leq 4, \quad a^2 + c^2 \leq 4. \tag{8}$$

From the form of coefficients of equation (7) it follows that the parameters b and c occur symmetrically in it. In the terms of the coefficients p_i we can separate out the factor abc so that a, b and c occur only in even powers, so the transformation $a \to -a$ leads to a distribution of number of equilibria symmetric with respect to either b - or c - axis. For the numerical investigation of equation (7) it is sufficient to consider the domain of the parameters delimited by inequalities (8) and the inequalities

$$-2 \leq a \leq 2, \quad 0 \leq b \leq 2, \quad 0 \leq c \leq 2; \tag{9}$$

in the plane (b, c) for the fixed the parameter a can be confined to the sector between the lines $c = b$ and $c = 0$ by virtue of symmetry. On the diagonal of the square $|b| \leq 2$, $|c| \leq 2$; when $|b| = |c|$, there is additional restriction $|b| \leq \sqrt{2}$.

We have analyzed numerically the dependence of the number of real roots of equation (7) in space of parameters delimited by inequalities (8), (9). We used classification results [6] of different distributions of number of real roots for values of parameters in the domain (8), (9). Here, using Computer Algebra applications, the analytic equations of the boundary surfaces between the regions with different number of equilibria were found. In this work both the classification of different distributions of number of equilibria and the coordinates of the bifurcations points combining analytical and numerical analysis were made.

For the fixed values of parameter a the number of positive real roots was determined in the plane (b, c) and domains with the fixed number of real roots were obtained. Using the parameter a in accordance with bifurcations points, mentioned above, we can obtain the numerical results with the same regions of constant number of equilibria. Critical values of the parameter a are the following

$$-2; \frac{-35+\sqrt{193}}{12}; -\frac{3}{2}; \frac{-25+\sqrt{73}}{12}; -\frac{7}{6}; -\frac{2+\sqrt{22}}{6}; -1; -\frac{39}{35}; -\frac{2}{3}; -\frac{1}{2}; \frac{2-\sqrt{22}}{6}; \frac{-2+\sqrt{22}}{6}; 2.$$

For the first critical value $a = -2$ the parameter values $b = c = 0$ are the only ones which correspond to equlibria (there are four of these). In the interval $-2 < a < \frac{-35+\sqrt{193}}{12}$ 8 equlibria are existed. For the next interval $\frac{-35+\sqrt{193}}{12} < a < -\frac{3}{2}$ 16 equlibria are existed, and after critical value $a = \frac{-25+\sqrt{73}}{12}$ until the value $a = 0$ there are domains of existence of 24 equlibria. As it was mentioned above, the number of equlibria to a positive value of a being symmetric with respect to either $b-$ or $c-$ axis.

Analysis of numerical results shows that 24 satellite equilibria exist for sufficiently small constant torques; as the magnitude of constant torque increases, the domain of existence of equilibria diminishes and when it is large enough, the satellite does not have a single equilibrium orientation.

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