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On supercritical flows around airfoil with shock waves

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Abstract

We consider planar steady-state flows of an ideal perfect gas around wing profiles. Subsonic free-stream velocity exceeds critical value that leads to the arising local supersonic zones probably containing shock waves. We establish some properties of supercritical flows in the assumption that the corresponding boundary value problem is well-posed in the class of discontinuous flows. We use a notion a "well-streamlined airfoil" as the synonym of a separation-free flow. We prove that the Chaplygin-Zhukovskii-Kutta condition is necessary and sufficient for the profile to be well-streamlined. We establish some asymptotical properties of the velocity in the neighborhood of the trailing edge of the profile. We prove that the shock wave cannot intersect the profile contour. We formulate an approach to designing weak-supercritical flows without shock waves.

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1. Introduction

We continue researching transonic (planar, steady-state, potential and vortical) flows of the air in the framework of the theory of ideal Clapeiron's gas [1].

The classical theory of the irrotational flows around airfoils based on the theory of analytical and pseudo-analytical functions is quite adequate at the moderate velocities of the flight, when the exterior flow is uniformly subsonic. But if subsonic velocity $|V_\infty|$ of the flight exceeds a critical value V_{cr} that depends on the profile P, the local supersonic zones appear in the flow. (Such flows are called supercritical.) Thus there is need in transonic gas dynamics based on the theory of differential equations of mixed elliptic-hyperbolic type.

A potential supercritical flow around the profile is known to be "unstable" with respect to continuous deformations of the contour: by virtue of well-known Nikolskii-Taganov theorem (see [2]), straightening small segment of the profile leads to discontinuity of the velocity field. Nevertheless it is possible to expect (especially when considering strictly convex profiles) that the mathematical problem is well posed in the class of the discontinuous flows, i.e. that straightening segment of small length results in arising shocks of small length or intensity.

Thus supercritical flows contain, as a rule, shock waves generating vorticity. The length and intensity of shocks increases along with the flight velocity. This results in sharply increasing energy losses. However, increasing the flight velocity reduces total losses. Therefore, the form of the “supercritical” wing profiles intended for the long-distance flights of civil aircraft is a subject to optimization. The represented results can be useful for designing supercritical lifting wing profiles of high efficiency.

2. Current definitions

Velocity $\boldsymbol{V}$, density ρ, temperature T and pressure p is assumed to be continuously differentiable functions of the Cartesian coordinates $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ in $\overline{\mathbf{R}^2\backslash P\backslash L}$, where L is lines of the shock waves that satisfy the Hugoniot conditions. The velocity vector $\boldsymbol{V}$ achieves a given value $\boldsymbol{V}_\infty$ at infinity.

By the definition given in the monograph [2], a wing profile P is a closed smooth curve, except for a “sharp-pointed trailing edge” with a small non-zero angle α.

By virtue of equation $div(\rho V) = 0$, there exists stream function $\psi(x, y)$. Its level lines that are called streamlines are vector lines of the velocity field. The stream line $\psi(x, y) = 0$ coincides with the profile contour. Two branching points $O_{1,2}$ divide the stream line $\psi = 0$ into the “upper” and “low” pieces.

Let us limit ourselves to the set of convex profiles.

Consider the straight line passing from the trailing edge such as the profile projection on this line be of maximal length. The intersection of the profile with this line is called the profile chord. The angle δ between the vector $\boldsymbol{V}_\infty$ and profile chord is called the attack angle.

Transonic aerodynamics (see the monograph [2]) says that for each profile P and the attack angle δ there exists a critical magnitude $V_{cr} = V_{cr}(P, \delta)$ of the subsonic velocity V_∞, when the subsonic velocity on the profile increases up to the sonic value. In this case the flow around profile is called critical. A set of the sonic points on the profile contour is being transformed at the further increase of V_∞ into supersonic sub-domain inside the flow domain. In this case the flow is called supercritical. The developed supersonic zones contain, as a rule, shock waves.

These shock waves in the supersonic zones, if they exist, generate vorticity that breaks equality between magnitudes of the velocity circulations on the profile and on infinity. This changes the lifting force and leads to emerging of the so called “wave resistance”

[1].

3. Condition for the ideal gas theory to be adequate

It is accepted in aerodynamics to describe laminar flows at large Re numbers using the theory of ideal gas taking into account an unseparated boundary layer.

However, if the boundary layer comes off and the developed areas of circular non-stationary flows are being formed, this theory becomes inadequate; in particular, the additional energy losses in these areas cannot be correctly calculated. A separation-free boundary layer is known to exist (see the work [3]) if the flow deceleration on the body is bounded by some number dependent on the flow data. Let us call a wing profile "well-streamlined" if this condition is fulfilled. Below only such wing profiles are considered.

Definition 3.1. A stationary flow around a profile P is separation-free, if one of two branching points $O_{1,2}$ of the stream line $\psi = 0$ is the sharp-pointed trailing edge O_2.

Actually this definition asserts sufficiency of the so called Zhukovskii-Chaplygin-Kutta condition for the profile to be well-streamlined. (Necessity will be proved in Theorem 2.) Initially, this condition has been formulated by the authors for incompressible fluids as a condition for the velocity to be continuous in closed exterior of the profile [4]. Indeed, if the condition is not fulfilled, the velocity of the incompressible fluid, which is bending around the sharp-pointed trailing edge, should turn out in this point into infinity.

S.A. Chaplygin has proved that this condition provides uniqueness of the potential flow of incompressible fluid with a given velocity on infinity. (Therefore, we shall call this Chaplygin's condition.) It has been established later that this condition guarantees uniqueness in more general case of the compressible gas as well [2]. However, if there are shock waves, the flow uniqueness is not proved up to now.

4. Velocity in the sharp trailing edge

It follows from the Chaplygin condition for the potential flows of incompressible fluid that $\boldsymbol{V} = 0$ in the sharp edge [4]. It is true also for the compressible fluid and for supercritical vortical flows, when vorticity arises due to shock waves.

Theorem 4.1. If a flow around a profile is separation-free and the Mach number is bounded, $\boldsymbol{V} = u\boldsymbol{i} + v\boldsymbol{j}$ is zero in the sharp-pointing trailing edge O_2.

Proof. In the air the formulas take place at $\gamma = 1,4$

$$T = T_0(\psi)\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1}, \rho = \rho_0(\psi)\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{\gamma-1}}, \\ M^2 = \frac{V^2}{\gamma R T_0(\psi) - (\gamma - 1)V^2/2} \tag{1}$$

Here M is Mach number, $\rho_0(\psi) = \rho|_{V=0} \neq 0$, $T_0(\psi) = T|_{V=0} \neq 0$ are the "stagnation" density and temperature,γ, R are adiabatic index and gas constant correspondingly. Let temperature T be finite constant $T_\infty = T|_{V=V_\infty}$ in incoming air. Then the stagnation temperature $T_0(\psi)$ in incoming air is a finite constant T_0, too. By virtue of the Hugoniot relation, T_0 does not change when streamlines intersects shocks, therefore $T_0(\psi) = T_0$ everywhere in the flow domain.

By virtue of Definition 3.1, the streamline $\psi = 0$ branches into the "upper" and "lower" segments of the profile contour. Therefore, the trailing edge O_2 should be a saddle singular point of the ordinary differential equation

$$\left.\frac{dy}{\partial x}\right|_P = \left.\frac{\rho(\rho_0^{\pm}(\psi)M)v}{\rho(\rho_0^{\pm}(\psi)M)u}\right|_P \Rightarrow \left.\frac{dy}{\partial x}\right|_{O_2} = \lim_{r \to r(O_2)} \frac{\rho(\rho_0^{\pm}(\psi)M)v}{\rho(\rho_0^{\pm}(\psi)M)u} \qquad (2)$$

The difference in the constant values $\rho_0^{\pm}(\psi)|_{\psi=0}$ of the stagnation density on the "upper" and "lower" sides of the profile can arise because of different intensity of the shock waves.

By condition, $M < \infty \Rightarrow |\boldsymbol{V}|^2 < 2RT_0\gamma/(\gamma-1) < \infty$, hence the right-hand part of (2) should have in O_2 the indeterminacy $v/u \sim 0/0$, consequently $\boldsymbol{V}|_{O_2} = 0$. Theorem 4.1 is proved. ■

5. Necessity of the Chaplygin condition

Theorem 5.1. If the Chaplygin condition is not fulfilled in the flow around the wing profile, then this flow is not separation-free.

Proof. Assume the contrary, then by Theorem 4.1, $\boldsymbol{V} \neq 0$ in the sharp-pointing trailing edge. If the flow is separation-free, the velocity vector should rotate in O_2 on the angle $\pi - \alpha$. This is impossible, if

$$\alpha < \alpha^* = \pi\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right) / \, 2, \;\; \alpha^*|_{\gamma=1,4} \approx 50^\circ$$

because if $\alpha = \alpha^*$, then the pressure on the low side of the profile will turn out into zero, hence the flow separation should arise. Indeed, the fluid bending the convex angle should at first accelerate up to sonic speed. After that the fluid should move satisfying asymptotic of the Prandtl-Meyer flow described by the equation of the characteristics in the hodograph-plane [5]

$$|\beta(M) - \beta(1)| = \sqrt{\frac{\gamma+1}{\gamma-1}} arctan \sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} -$$

$$-arctan\sqrt{M^2-1} \leq \pi - \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)\Bigg|_{\gamma=1,4} = 180^\circ - 50^\circ = 130^\circ$$

where $\beta(M)$ is the velocity argument.

Note that as this formula takes place along the stream line $\psi = 0$, this reasoning is valid for vortical flows as well.

We omit the proof of impossibility of the case $\alpha \geq 50°$, as this one is not used in the aviation. Theorem 5.1 is proved. ■

6. Asymptotic behavior of the velocity in neighborhood of the sharp trailing edge

It is proved for incompressible fluids with use of the analytic functions theory that the velocity in the potential flow has the power singularity in the vicinity of the sharp edge (see [4])

$$V = C|z|^{\alpha/2\pi-\alpha} + \cdots \tag{3}$$

Theorem 6.1. If a flow around the profile is separation-free, the main term of the potential velocity in the vicinity of the trailing edge O_2 is expressed by formula (3).

Proof. Consider the mapping of the profile exterior into the hodograph-plane (V, β). The stream function $\psi(V, \beta)$ of the potential flow satisfies the Chaplygin equation [6]

$$\psi_{VV} + \psi_V \frac{1 + M^2(V)}{V} + \psi_{\beta\beta} \frac{1 - M^2(V)}{V} = 0 \tag{4}$$

S.A. Chaplygin has proved (see the monograph [6]) that $\psi(V, \beta)$ is expressed by the series that uniformly converges at $V^2 < 2\gamma RT_0/(\gamma + 1)$, i.e. in the subsonic strip of the hodograph-plane (V, β)

$$\psi(V, \beta) = \sum_{n=0}^{\infty} c_n \tau^{n/2} F(a_n, b_n, n+1, \tau) \sin n\beta, \quad \tau = \frac{V^2}{2RT_0/(\gamma - 1)}$$

Here $F(a_n, b_n, n+1, \tau)$ is the hyper-geometric function.

Therefore, the formula is valid in vicinity of the point $V = 0$

$$\psi(V, \beta) = \sum_{n=0}^{\infty} c_n V^n (1 + O(V^2)) \sin n\beta \tag{5}$$

(Compare eq. (5) with the formula $\psi(V, \beta) = \sum_{n=0}^{\infty} c_n V^n \sin n\beta$ expressing the stream function in incompressible fluid.)

The neighborhood of the point O_2 in the hodograph-plane (V, β) belongs to the strip $0 \leq \beta \leq \alpha$. Hence the image of the streamline $\psi = 0$ that goes out from the profile is situated inside this strip. Therefore, we have that $n = 2\pi m/\alpha,\ m = 1, 2, \ldots$ and the streamline $\psi = 0$ has in the point O_2 the tangent, which is the same as the bisectrix of the angle that is formed by the tangents to the velocity vectors on the upper and lower profile sides. Thus formula (5) is transformed to the form

$$\psi(V,\beta)=\sum_{m=0}^{\infty} c_m V^{\frac{2m\pi}{\alpha}}(1+O(V^2))\sin\frac{2m\pi}{\alpha}\beta \tag{6}$$

To pass into the physical plane, the formulas should be used

$$\begin{aligned} x_V &= -\frac{\psi_V \sin\beta + V^{-1}(1-M^2(V))\psi_\beta\cos\beta}{V\rho(V)},\ x_\beta=\frac{V\psi_V\cos\beta-\psi_\beta\sin\beta}{V\rho(V)}\\ y_V &= \frac{\psi_V\cos\beta - V^{-1}(1-M^2(V))\psi_\beta\sin\beta}{V\rho(V)},\ y_\beta=\frac{\cos\beta\psi_\beta+V\sin\beta\psi_\beta}{V\rho(V)} \end{aligned} \tag{7}$$

Differentiating the uniformly converged series (6), we obtain

$$\begin{aligned} \psi_V &= \left[V^{\frac{2\pi}{\alpha}-1}+O\left(V^{\frac{2\pi}{\alpha}+1}\right)\right]\sin\frac{2\pi}{\alpha}\beta+O\left(V^{\frac{4\pi}{\alpha}-1}\right)\sin\frac{4\pi}{\alpha}\beta\\ \psi_\beta &= \left[V^{\frac{2\pi}{\alpha}}+O\left(V^{\frac{2\pi}{\alpha}+2}\right)\right]\cos\frac{2\pi}{\alpha}\beta+O\left(V^{\frac{4\pi}{\alpha}}\right)\cos\frac{4\pi}{\alpha}\beta \end{aligned}$$

Substituting these expressions into formulas (7) and taking into account that

$$M^2(V)\underset{V\to 0}{\sim}V^2,\ \ \rho(V)\underset{V\to 0}{\sim}\rho_0(1-O(V^2))\ \ \rho_0|_{\psi=0}=const,$$

we obtain asymptotical formulas

$$\begin{aligned} x_V &= V^{\frac{2\pi}{\alpha}-2}\sin\beta\sin\frac{2\pi\beta}{\alpha}+V^{\frac{2\pi}{\alpha}}\cos\beta\cos\frac{2\pi\beta}{\alpha}+\dots=O\left(V^{\frac{2\pi}{\alpha}-2}\right)+\dots,\\ y_V &= V^{\frac{2\pi}{\alpha}-2}\cos\beta\sin\frac{2\pi\beta}{\alpha}-V^{\frac{\pi}{2\alpha}}\sin\beta\cos\frac{2\pi}{\alpha}\beta+\dots=O\left(V^{\frac{2\pi}{\alpha}-2}\right)+\dots,\\ x_\beta &= V^{\frac{2\pi}{\alpha}-1}\left(\cos\frac{2\pi}{\alpha}\beta\cos\beta-\sin\beta\cos\frac{2\pi}{\alpha}\right)+\dots=O\left(V^{\frac{2\pi}{\alpha}-1}\right)+\dots,\\ y_\beta &= V^{\frac{\pi}{2\alpha}-1}\left(\sin\beta\cos\beta+\sin\beta\sin\frac{2\pi}{\alpha}\beta\right)+\dots=O\left(V^{\frac{2\pi}{\alpha}-1}\right)+\dots, \end{aligned}$$

Integrating these expressions, we obtain

$$|z|=|x+iy|\underset{V\to 0}{=}C_1V^{(2\pi-\alpha)/\alpha}+\cdots\ \Rightarrow\ V\underset{z\to O_2}{=}C_2|z|^{\alpha/(2\pi-\alpha)}+\cdots$$

Theorem 5.1 is proved. ■

7. Shock wave inside supersonic region

Theorem 7.1. If the Chaplygin condition is fulfilled in a supercritical flow around a smooth strictly convex profile of bounded curvature, then there are no shock waves intersecting the profile contour.

Proof. A point of the shock wave, in which its intensity equals zero, is called the endpoint. If the flow velocity in the endpoint is supersonic, then it cannot belong to the profile contour. Indeed, the endpoint is a cusp of the envelope of characteristics of one family that is continuously continued into interior of the profile, therefore the velocity gradient is infinite there. But by condition, the contour curvature is bounded. Hence this endpoint cannot belong to the profile contour.

If the endpoint of the shock wave is sonic, then this one should be orthogonal in this point to the profile contour. This is impossible due to non-zero of the contour curvature.

Consider now the case when a shock wave of non-zero intensity intersects the profile contour. Consider firstly the case of asymmetric profile relative to the profile chord.

As the contour is smooth, the shock wave should be orthogonal to the contour. Therefore, the total pressure after the shock is less than total pressure in the incoming flow. By virtue of the profile asymmetry, the total pressures on two segments of the streamline $\psi = 0$ divided by the points $O_{1,2}$ are different in general case. Therefore the joint streamline that goes from the point O_2 should be a contact break. But this is impossible, as by virtue of Theorem **??**, $V = 0$ in the trailing edge. Therefore, the static pressures on the different shores of this streamline do not coincide because of the total pressure distinction. Theorem 7.1 is proved for an asymmetric profile. ■

Strictly speaking, the above proof is non-applicable to symmetric flow around a symmetric profile. However, assuming continuous dependence of the separation-free flow on the attack angle, we obtain the same result for the symmetric wing as well.

8. Designing weakly supercritical flows without shocks

When increasing the stream velocity up to the critical one, there is a limit of the subcritical flows sequence that is called the "critical flow" (see the monograph [2]).

Let $\delta V_\infty = V_\infty - V_{cr}$. Remembering the well-known Ringleb solution [6], let us assume that at small δV_∞ there exists a supercritical separation-free flow without shocks that depends continuously on δV_∞. It is extremely important to learn to design the airfoils intended for realization of such flows.

In our opinion, to solve this problem a modification of the method [7] can be successful.

(Using the Chaplygin hodograph transformation, we have developed in [7] a well-posed numerical method for designing sub- and critical lifting profiles that are intended for cruiser flights with high subsonic velocity. A singular Dirichlet problem for stream function satisfying the Chaplygin equation on two-sheet Riemann surface in subsonic part of the hodograph-plane was formulated. The lifting wing profiles of 8–10% thickness for $M_\infty = 0,8 - 0,85$ were calculated.) The mentioned modification is the problem "in variations" at small δV_∞ determined on the hodograph-domain of a critical flow. The above Dirichlet problem changes into the problem, in which the boundary value $\psi = 0$ is given as before on the subsonic part of the hodograph-contour and the derivative $\partial\psi/\partial V = f(\beta)$ is given instead of the condition $\psi = 0$ on a segment of the sonic line. The theory of equations of the degenerated elliptic type [8] testifies on unique solvability of this problem at proper limitations. It should be noted that this solution being transformed into the physical plane, most likely, will not be continuously depending on small computational errors in the class of potential flows. Nevertheless, taking into account stability of the real flows, one can assume that these solutions will be well-posed in the more wide class of the vortical flows with small shock waves. However, to check this there is need to solve the direct problem of the flow around a given profile in the class of vortical flows without smearing shock waves. This is possible, apparently, only when using equation for the stream function.

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