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COMPUTER ALGEBRA

Symbolic–Numerical Methods of Studying Equilibrium Positions of a Gyrostat Satellite

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Abstract—Methods of computer algebra are used to study properties of a nonlinear algebraic system that determines equilibrium positions of a gyrostat satellite moving along a circular orbit. Bifurcation values of parameters such that the number of equilibrium positions changes are found numerically. Detailed numerical analyses of evolution of existence domains of different numbers of equilibria in the space of dimensionless parameters are performed.

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1. INTRODUCTION

Celestial mechanics is one of the popular domains of application of symbolic computation methods. In astrodynamics computer algebra methods are much less common. Problems of orientation of artificial Earth satellites are one of the biggest sections of astrodynamics. Designing orientation systems for Earth artificial satellites is one of the important directions of development of space engineering. Satellite orientation can be done with the help of both active and passive methods. Design of passive orientation systems can employ properties of the gravitational and magnetic fields, atmospheric drag, solar radiation pressure, gyroscopic properties of rotating bodies, etc. Passive orientation systems possess an important property, viz. they can function for a long time without energy consumption. Gravitational systems are most common of all passive orientation systems. Their operation is based on the fact that the satellite with unequal principal central moments of inertia in the central Newtonian force field has 24 equilibrium positions at the circular orbit, including 4 stable positions [1]. With rotors rotating with the constant angular velocity with respect to the satellite body added to its design, one can obtain new, more complex equilibrium positions of the gyrostat satellite that are interesting in terms of practical application. A great number of works deal with the problem of determining equilibrium positions of a gyrostat satellite. Dynamics of satellites with gravitational orientation systems are considered in detail in [1]. In [2–6], all equilibrium positions of a gyrostat were determined for some special cases where the vector of gyrostatic moment is parallel to one of the central axes of inertia of the gyrostat satellite or is in one of the planes formed by the principal central axes of inertia.

The general case of the gyrostat problem was first considered in [7], where the results of symbolic study of dynamics of a satellite subjected to the gravitational and gyrostatic moments are given. In this work, we propose a method for determining all equilibrium positions of the gyrostat satellite for given values of the vector of gyrostatic moment and principal central moments of inertia, which based on the algorithm of constructing Groebner bases and the concept of resultant, and obtain their existence conditions depending on four dimensionless parameters of the system. We found bifurcation values of the parameters for which the number of equilibrium positions changes. We performed detailed numerical analysis of evolution of existence domains of different numbers of equilibria in the space of the dimensionless parameters.

Symbolic-numerical methods of determining equilibrium positions presented in this work have been successfully applied before to analyze aerodynamic and gravitational forces affect equilibrium orientations of the satellite [8].

2. EQUATIONS OF MOTION

Consider the problem of rotary motion of a gyrostat satellite (a satellite or gyrostat, in what follows) that is a solid body with statically and dynamically balanced rotors placed inside it. We consider the angular velocities of rotation of rotors with respect to the satellite body to be constant and the center of mass of the gyrostat satellite to be moving at a circular orbit.

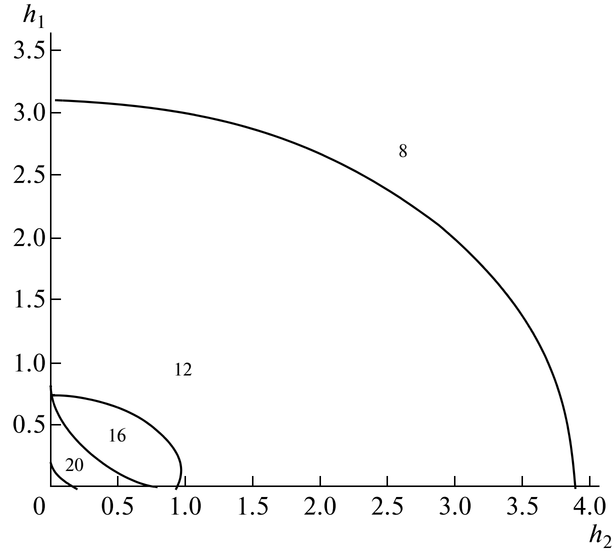

Fig. 1. $\nu = 0.2$, $h_3 = 0.01$.

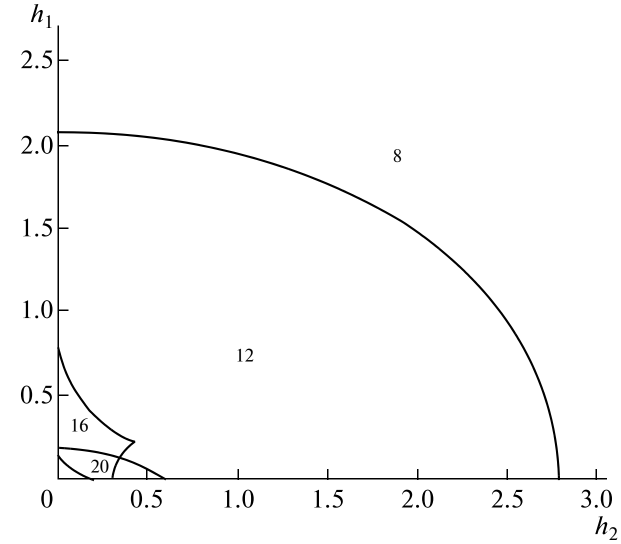

Fig. 2. $\nu = 0.2$, $h_3 = 0.4$.

To write down the equations of motion, we introduce two right Cartesian systems of coordinates with the coordinate origin at the center of masses *O*. *OXYZ* is the orbital system of coordinates. The *OZ* axis is directed along the radius vector that connects the center of mass of the Earth and the gyrostat; the *OX* axis is directed along the vector of the linear velocity of the center of mass *O*. *Oxyz* is the system of coordinates associated with the gyrostat satellite; *Ox*, *Oy*, and *Oz* are the principal central axes of inertia of the gyrostat satellite. We use the Euler angles ψ, ϑ, and φ to determine orientation of the coordinate system *Oxyz* with respect to the orbital system of coordinates. The direction cosines of the axes of the system of coordinates *Oxyz* in the orbital system of coordinates *OXYZ* can be expressed via Euler angles using the relations [7]

$$\begin{aligned}
a_{11} &= \cos(x, X) = \cos\psi\cos\varphi \\
&\quad - \sin\psi\cos\vartheta\sin\varphi, \\
a_{12} &= \cos(y, X) = -\cos\psi\sin\varphi \\
&\quad - \sin\psi\cos\vartheta\cos\varphi, \\
a_{13} &= \cos(z, X) = \sin\psi\sin\vartheta, \\
a_{21} &= \cos(x, Y) = \sin\psi\cos\varphi \\
&\quad + \cos\psi\cos\vartheta\sin\varphi, \\
a_{22} &= \cos(y, Y) = -\sin\psi\sin\varphi \\
&\quad + \cos\psi\cos\vartheta\cos\varphi, \\
a_{23} &= \cos(z, Y) = -\cos\psi\sin\vartheta, \\
a_{31} &= \cos(x, Z) = \sin\vartheta\sin\varphi, \\
a_{32} &= \cos(y, Z) = \sin\vartheta\cos\varphi, \\
a_{33} &= \cos(z, Z) = \cos\vartheta.
\end{aligned} \tag{1}$$

Then, we can write the equations of motion of the gyrostat satellite with respect to its center of mass as follows [1, 7]:

$$\begin{aligned}
&A\dot{p} + (C - B)qr - 3\omega_0^2(C - B)a_{32}a_{33} \\
&\quad - \bar{H}_2 r + \bar{H}_3 q = 0, \\
&B\dot{q} + (A - C)rp - 3\omega_0^2(A - C)a_{33}a_{31} \\
&\quad - \bar{H}_3 p + \bar{H}_1 r = 0, \\
&C\dot{r} + (B - A)pq - 3\omega_0^2(B - A)a_{31}a_{32} \\
&\quad - \bar{H}_1 q - \bar{H}_2 p = 0,
\end{aligned} \tag{2}$$

$$\begin{aligned}
p &= \dot{\psi}a_{31} + \dot{\vartheta}\cos\varphi + \omega_0 a_{21} = \bar{p} + \omega_0 a_{21}, \\
q &= \dot{\psi}a_{32} - \dot{\vartheta}\sin\varphi + \omega_0 a_{22} = \bar{q} + \omega_0 a_{22}, \\
r &= \dot{\psi}a_{33} + \dot{\varphi} + \omega_0 a_{23} = \bar{r} + \omega_0 a_{23}.
\end{aligned} \tag{3}$$

In Eqs. (2) and (3), *A*, *B*, and *C* are the principal central moments of inertia of the gyrostat, *p*, *q*, *r* and $\bar{H}_1$, $\bar{H}_2$, $\bar{H}_3$ are projections of the absolute angular velocity and projections of the vector of gyrostatic moment of the gyrostat onto the axes *Ox*, *Oy*, and *Oz*, and ω_0 is the angular velocity of the center of mass of the gyrostat moving at a circular orbit. The dot stands for differentiation with respect to *t*.

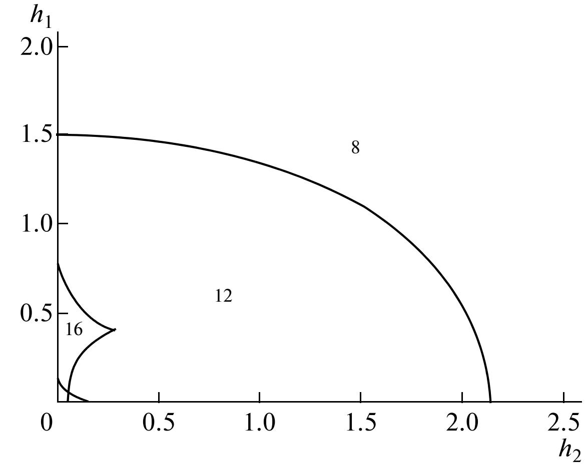

Fig. 3. $\nu = 0.2$, $h_3 = 0.8$.

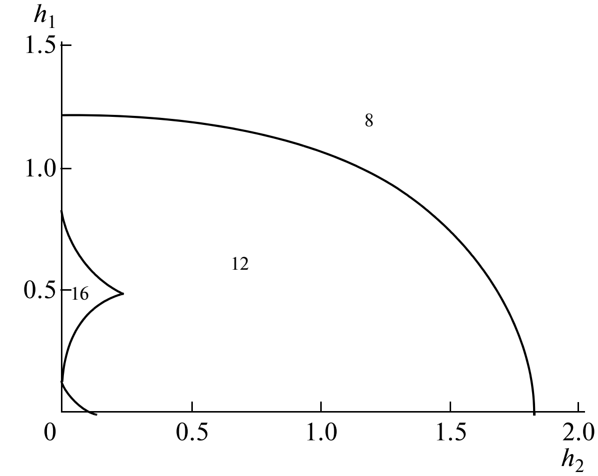

Fig. 4. $\nu = 0.2$, $h_3 = 1.048$.

3. EQUILIBRIUM POSITIONS OF THE GYROSTAT SATELLITE

Setting $\psi = \psi_0 = \text{const}$, $\vartheta = \vartheta_0 = \text{const}$, $\varphi = \varphi_0 = \text{const}$ and $H_1 = \overline{H}_1/\omega_0$, $H_2 = \overline{H}_2/\omega_0$, $H_3 = \overline{H}_3/\omega_0$ in (2) and (3), we obtain, when $A \neq B \neq C$, the equations

$$(C-B)(a_{22}a_{23} - 3a_{32}a_{33}) = H_2a_{23} - H_3a_{22},$$
$$(A-C)(a_{23}a_{21} - 3a_{33}a_{31}) = H_3a_{21} - H_1a_{23}, \quad (4)$$
$$(B-A)(a_{21}a_{22} - 3a_{31}a_{32}) = H_1a_{22} - H_2a_{21},$$

that help us to find equilibrium positions of the gyrostat satellite in the orbital coordinate system. For further study, the equivalent system

$$Aa_{11}a_{31} + Ba_{12}a_{32} + Ca_{13}a_{33} = 0,$$
$$Aa_{11}a_{21} + Ba_{12}a_{22} + Ca_{13}a_{23}$$
$$+ (H_1a_{11} + H_2a_{12} + H_3a_{13}) = 0, \quad (5)$$
$$4(Aa_{21}a_{31} + Ba_{22}a_{32} + Ca_{23}a_{33})$$
$$+ (H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0,$$

which is obtained by projecting (4) onto the axes of the orbital coordinate system is more convenient to use. Substituting the expression for the direction cosines from (1) into (4) or (5), we arrive at the system of three equations in the unknowns ψ, ϑ, and φ.

Another, more convenient, way to close Eqs. (5) is to add six conditions of orthogonality of the direction cosines

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1,$$
$$a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0,$$
$$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1, \quad (6)$$
$$a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0,$$
$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1,$$
$$a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0.$$

Equations (5) and (6) form the closed algebraic system of equations with respect to the nine unknown direction cosines that define the equilibrium positions of the gyrostat. For the system of equations (5) and (6), we state the following problem: for given A, B, C, H_1, H_2, and H_3, find all nine direction cosines, i.e., all equilibrium positions of the gyrostat. The system of equations (5) and (6) in nine unknowns was solved for some special cases. When the vector of gyrostatic moment is parallel to one of the principal central axes of inertia of the gyrostat satellite, say, Oy, given $H_1 = 0$, $H_2 \neq 0$, $H_3 = 0$ [2, 3, 6], all equilibrium positions were found analytically depending on two dimensionless parameters of the problem and sufficient conditions of stability of these equilibrium positions were obtained in the form of simple inequalities. The solution of the problem when the vector of gyrostatic moment is parallel to the plane of any two principal central axes of inertia, say, Oxz ($H_1 \neq 0$, $H_2 = 0$, $H_3 \neq 0$) is obtained in [4, 5].

For $H_1 = H_2 = H_3 = 0$, system (5), (6) was proved to have 24 solutions, which determine equilibrium orientations of the satellite that is a solid body [1].

In this work, we study the equilibrium positions of the gyrostat for the general case when $H_1 \neq 0$, $H_2 \neq 0$, $H_3 \neq 0$. The general case of the problem was first considered in [7]. As shown in [7], the system of equations (5), (6) can be resolved with respect to a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, and a_{23} as follows

$$a_{11} = 4(C-B)a_{32}a_{33}/F,$$
$$a_{21} = 4(I_3 - A)a_{31}/F,$$

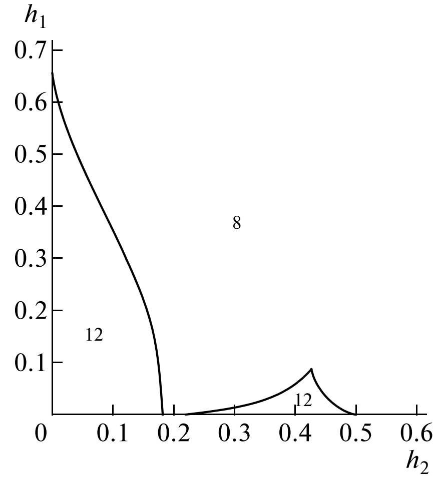

Fig. 5. $\nu = 0.2$, $h_3 = 3.264$.

$$a_{12} = 4(A-C)a_{33}a_{31}/F,$$
$$a_{22} = 4(I_3-B)a_{32}/F, \tag{7}$$
$$a_{13} = 4(B-A)a_{31}a_{32}/F,$$
$$a_{23} = 4(I_3-C)a_{33}/F.$$

Here, $F = H_1a_{31} + H_2a_{32} + H_3a_{33}$, $I_3 = Aa_{31}^2 + Ba_{32}^2 + Ca_{33}^2$.

To solve algebraic system (5), (6), we applied the algorithm of constructing the Groebner bases [9]. The method of constructing the Groebner basis is an algorithmic procedure for complete reduction of the problem in the case of the system of polynomials in many variables to the polynomial in one variable. Using the Groebner[gbasis] Maple package [10] for constructing Groebner bases with linear ordering with respect to *tdeg* powers, we constructed the Groebner basis for system of nine polynomials (5), (6) with nine variables a_{ij} $(i, j = 1, 2, 3)$. Below are the polynomials from the constructed Groebner basis that depend only on three variables a_{31}, a_{32}, and a_{33}

$$16[(B-C)^2a_{32}^2a_{33}^2 + (C-A)^2a_{33}^2a_{31}^2 + (A-B)^2a_{31}^2a_{32}^2]$$
$$= (H_1a_{31} + H_2a_{32} + H_3a_{33})^2(a_{31}^2 + a_{32}^2 + a_{33}^2),$$
$$4(B-C)(C-A)(A-B)a_{31}a_{32}a_{33} \tag{8}$$
$$+ [H_1(B-C)a_{32}a_{33} + H_2(C-A)a_{33}a_{31} + H_3(A-B)a_{31}a_{32}]$$
$$\times (H_1a_{31} + H_2a_{32} + H_3a_{33}) = 0,$$

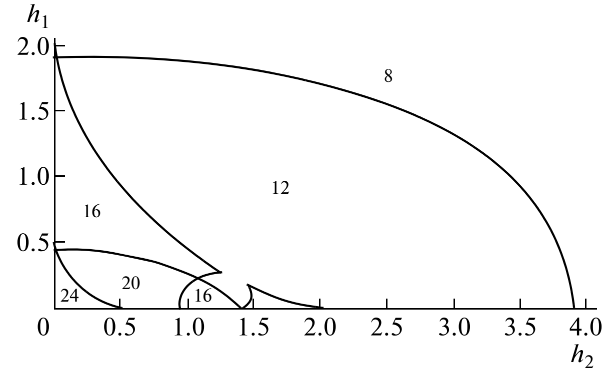

Fig. 6. $\nu = 0.5$, $h_3 = 0.01$.

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1.$$

One fails to obtain an algebraic equation of one unknown in the Groebner basis since the number of parameters is great. Nevertheless, taking into account that the first two equations in (8) are homogeneous and moving to new unknowns $x = a_{31}/a_{33}$, $y = a_{32}/a_{33}$, $h_i = H_i/(B-C)$, $\nu = (B-A)/(B-C)$, we obtain the following algebraic system of equations with respect to the variables x and y:

$$a_0y^2 + a_1y + a_2 = 0,$$
$$b_0y^4 + b_1y^3 + b_2y^2 + b_3y + b_4 = 0, \tag{9}$$

where

$$a_0 = h_2(h_1 - \nu h_3x),$$
$$a_1 = h_1h_3 + [4\nu((1-\nu) + h_1^2 - (1-\nu)h_2^2 - \nu h_3^2]x - \nu h_1h_3x^2,$$
$$a_2 = -(1-\nu)h_2(h_1x + h_3)x,$$
$$b_0 = h_2^2,$$
$$b_1 = 2h_2(h_1x + h_3),$$
$$b_2 = (h_2^2 + h_3^2 - 16) + 2h_1h_3x + (h_1^2 + h_2^2 - 16\nu^2)x^2,$$
$$b_3 = 2h_2(h_1x + h_3)(1 + x^2),$$
$$b_4 = (h_1x + h_3)^2(1 + x^2) - 16(1-\nu)^2x^2.$$

Using the concept of resultant to eliminate y from system (9), we obtain, by symbolic computations of the determinant of the resultant matrix, the algebraic equation of the twelfth order with respect to x

$$p_0x^{12} + p_1x^{11} + p_2x^{10} + p_3x^9 + p_4x^8 + p_5x^7$$

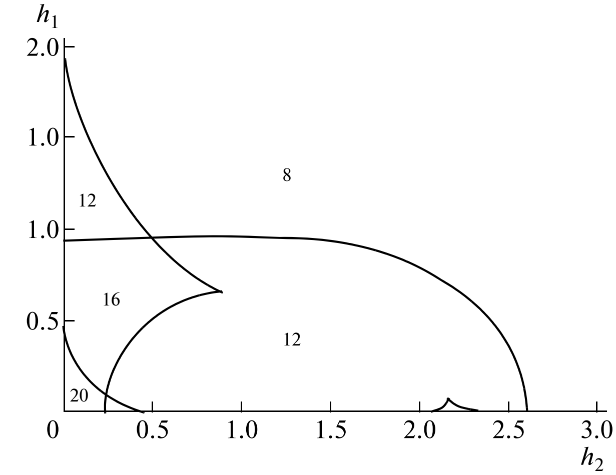

Fig. 7. $\nu = 0.5$, $h_3 = 0.5$.

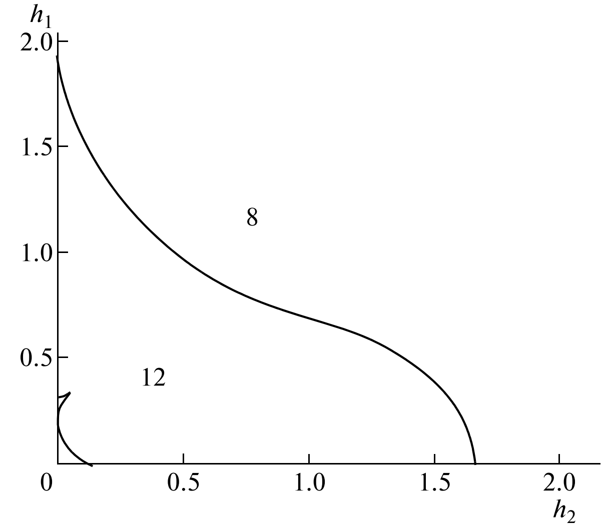

Fig. 8. $\nu = 0.5$, $h_3 = 1.182$.

$$+ p_6x^6 + p_7x^5 + p_8x^4 + p_9x^3 + p_{10}x^2 \tag{10}$$
$$+ p_{11}x + p_{12} = 0,$$

with its coefficients being the eighth-order polynomials of the parameters of the system h_1, h_2, and h_3:

$$p_0 = -h_1^4h_3^4\nu^6,$$

$$p_1 = 2h_1^3h_3^3\nu^5[2h_1^2 - h_2^2(\nu - 1) - 2\nu(h_3^2 + 2\nu - 2)], \ldots$$
$$p_{11} = -2h_1^3h_3^3[(2h_1^2 - h_2^2(\nu - 1) - 2\nu(h_3^2 + 2\nu - 2)], \tag{11}$$

$$p_{12} = -h_1^4h_3^4.$$

Expressions for the coefficients of Eq. (10) are quite bulky and occupy more than two text pages (78 lines), which is why we do not give them here since the volume of this work is limited. The coefficients of Eq. (10) are given in full form in the preprint [12].

The number of real roots of the algebraic equation obtained is even and not greater than 12. Substituting the value of the real root x_1 of algebraic equation (10) into the equations of system (9), we find the coinciding root y_1 of these equations. For each solution x_1 and y_1, one can find two values of a_{33} from the last equation of system (8) and, then, their respective values a_{31} and a_{32}. Thus, each real root of the algebraic equation is matched with two sets of values a_{31}, a_{32}, a_{33} that, by (7), unambiguously determine all other direction cosines a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}. Hence, the gyrostat satellite at the circular orbit can have no more than 24 equilibrium positions.

Using Eqs. (10), (11) together with systems (7)–(9), one can find all equilibrium positions of the gyrostat satellite for given values of parameters.

4. NUMERICAL METHODS OF STUDYING EQUILIBRIA

To study equilibrium positions of the gyrostat satellite, we state the problem of finding domains with the same numbers of real roots of Eq. (10) in the space of parameters. We considered the possibility to decompose the space of parameters into domains with the same numbers of real roots by symbolic computation of the discriminant hyper surface given by the discriminant of polynomial (10). Since expressions for the coefficients of polynomial (10) are bulky, it is impossible to study symbolically the system of algebraic equations that define the sets of singular points of the discriminant hyper surface because it takes several hundreds of symbolic lines to write this system down.

We used Mathematica 8.0 [11] to study the number of real solutions of Eq. (10) depending on the values of the parameters. Without loss of generality, we can perform numerical study given that $B > A > C$; then, $0 < \nu < 1$. The projections of the gyrostatic moment h_1, h_2, and h_3 can take any non-zero values. Coefficients of Eq. (10) depend on four dimensionless parameters ν, h_1, h_2, and h_3; the equations of original system (5) include six parameters. Reduction of the number of parameters is critical for the numerical study.

Now, consider properties of algebraic equation (10) in more detail. It follows from the form of the coefficients of Eq. (10) that the number of real roots does not depend on of the signs of the parameters h_1, h_2, and h_3. Indeed, the parameters h_1, h_2, and h_3 are to even powers only in the expressions for the coefficients of Eq. (10) of the even powers of x while the coefficients of the odd powers of x are h_1h_3 multiplied by the factor that only depends on even powers of h_1, h_2, and h_3. Hence, when the signs of the parameters h_1, h_2, and h_3 change, only the sign of the product h_1h_3 can change

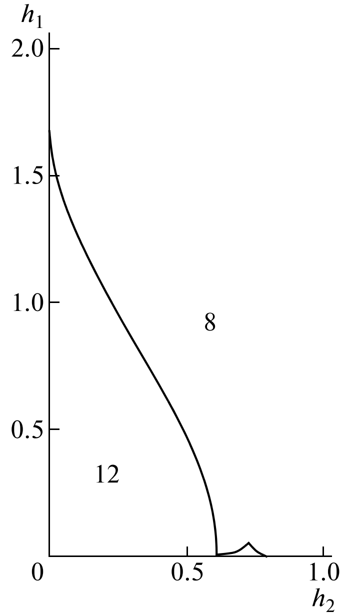

Fig. 9. $\nu = 0.5$, $h_3 = 2.412$.

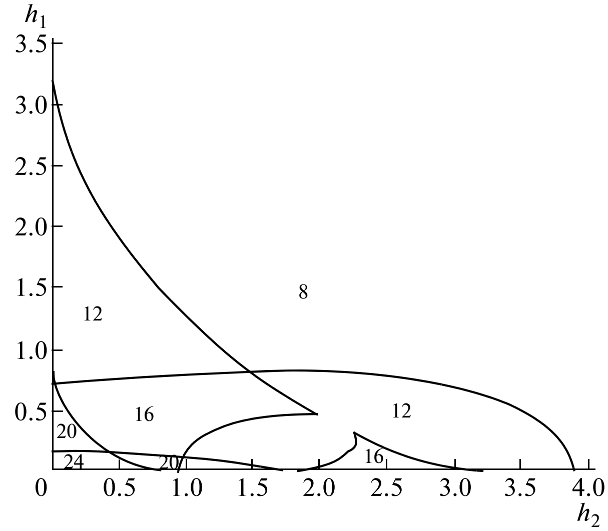

Fig. 10. $\nu = 0.8$, $h_3 = 0.01$.

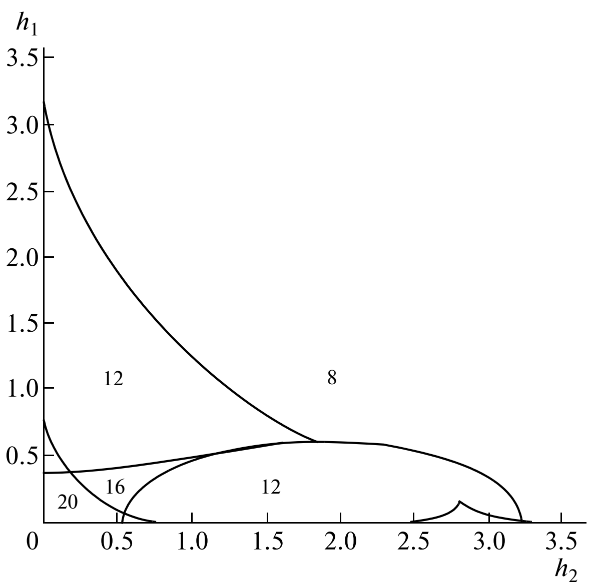

Fig. 11. $\nu = 0.8$, $h_3 = 0.2$.

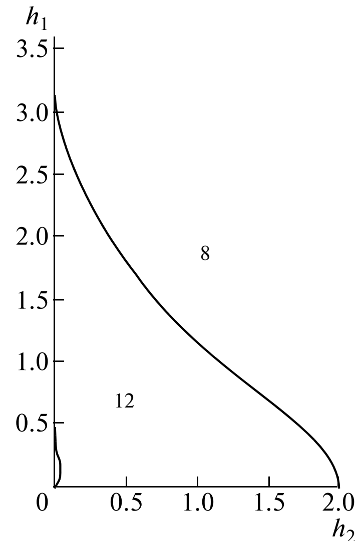

Fig. 12. $\nu = 0.8$, $h_3 = 0.909$.

and, thus, so does the sign of the real roots of Eq. (10). The absolute value and the number of real roots remains the same.

We analyzed the number of real roots of Eq. (10) numerically for positive values h_1, h_2, and h_3 and $0 < \nu < 1$. We performed calculations at the nodes of a uniform grid on the plane (h_1, h_2) for fixed values of ν and h_3. We found numerically the boundary points at which the number of real roots changes. In fact, we calculated the two-dimensional section of the discriminant hyper surface that is given implicitly by the algebraic equation of two parameters $g(h_1, h_2) = 0$. Experiments showed that, to obtain smooth boundary curves, one needs to perform calculations with the step 0.0001. Computations with such accuracy become very laborious. Indeed, roots at 10^9 nodes are to be calculated for a 4×4 domain on the plane (h_1, h_2). This is why we performed calculations in two stages. At the first stage, we calculated the number of real roots of Eq. (10) at 10^7 nodes with the step 0.001. At the second stage, the number of real roots was calculated in the neighborhood of the approximately calculated boundary between the domains with the constant numbers of real roots at the nodes of the grid with the step 0.0001. Further, for fixed values h_2, we found the variable of boundary points h_1 up to the given accuracy by the dichotomy method applied to the computation segment and implemented as the Mathematica package. Numerical methods of solving equations implemented in Mathematica help to find their roots up to the given accuracy. This advantage of Mathematica allowed us to calculate roots of the algebraic equations for very small values of coefficients not exceeding 10^{-48} when the values of parameters h_1 and h_2 were taken to be 10^{-6}.

Taking into account that Eq. (10) was obtained for $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$, we performed calculations with increased accuracy with the step 0.000001 in the neighborhood of zero values of the parameters h_1, h_2, and h_3.

We performed calculations for the values $\nu = 0.01$, $\nu = 0.1$, $\nu = 0.2$, $\nu = 0.3$, $\nu = 0.4$, $\nu = 0.5$, $\nu = 0.6$, $\nu = 0.7$, $\nu = 0.8$, $\nu = 0.9$, and $\nu = 0.99$.

Figures 1–13 show the results of calculations of evolution of boundaries between the domains with the same numbers of real roots on the plane (h_1, h_2) for the values $\nu = 0.2$ (the value close to the axially symmetric case of $\nu = 0$ or $A = B$), $\nu = 0.5$, $\nu = 0.8$ (the value close to the axially symmetric case of $\nu = 1$ or $A = C$). In [12], the boundaries between the domains with the constant numbers of real roots are shown to be given by equations of circles for axially symmetric case of $\nu = 0$ or $A = B$, while the boundaries between the domains with the same numbers of real roots are given by the equations of astroids for axially symmetric case of $\nu = 1$ or $A = C$.

Analysis of numerical results shows that, as the value of the parameter h_3 grows, the domains with the constant numbers of real roots become smaller and, eventually, disappear completely. The points at the space of parameters starting from which the domains with certain numbers of real roots disappear are called bifurcation points. The table presents results of calculation of bifurcation values of parameters obtained in [13].

Bifurcation values of ν and h_3

ν	$h_3(24/20)$	$h_3(20/16)$	$h_3(16/12)$	$h_3(12/8)$
0.01	0.99	0.999	3.959	4.0
0.1	0.90	1.021	3.610	4.0
0.2	0.80	1.048	3.264	4.0
0.3	0.70	1.082	2.950	4.0
0.4	0.60	1.124	2.669	4.0
0.5	0.50	1.182	2.412	4.0
0.6	0.40	1.186	2.167	4.0
0.7	0.30	1.105	1.915	4.0
0.8	0.20	0.909	1.629	4.0
0.9	0.10	0.676	1.245	4.0
0.99	0.01	0.168	0.997	4.0

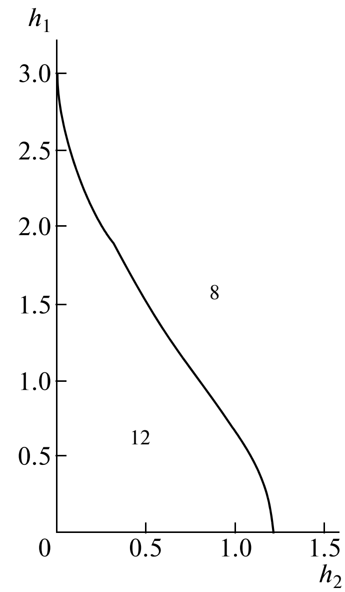

Fig. 13. $\nu = 0.8$, $h_3 = 1.629$.

It follows from the table that the bifurcation value of the parameter h_3, for which the domains with 24 equilibrium solutions (12 real roots) vanish, varies as $h_3 = 1 - \nu$. For the domains with the number of equilibria equal to 20 (10 real roots), the bifurcation value of the parameter h_3 grows as the parameter ν increases up to the value $\nu = 0.6$ and decreases as ν continues to grow. For the domains with the number of equilibria equal to 16 (8 real roots), the bifurcation value of the parameter h_3 decreases as the parameter ν grows. The domains with the number of equilibria equal to 12 become smaller as the value of the parameter h_3 grows. The central part of these domains vanishes in the neighborhood of the origin for $h_3 = 4$. For $h_3 \geq 4$, there exist small domains with the number of equilibria equal to 12 located near the axis Oh_2 with the typical sizes along the Oh_1 and Oh_2 axes not exceeding 10^{-1}. As the value of h_3 grows, these domains become smaller and move to the right along the Oh_2 axis.

Consider the nature of change of the domains with the number of equilibria equal to 24, 20, 16, 12, and 8 in more detail when $\nu = 0.2$ (Figs. 1–5).

Analysis of numerical results shows that, for $\nu = 0.2$, the domains with the number of equilibria equal to 24, 20, 16, 12, and 8 exist in the plane (h_1, h_2) for $h_3 < 0.8$ (Figs. 1 and 2). In Fig. 1 ($\nu = 0.2$, $h_3 = 0.01$), the domain with the number of equilibria equal to 24 is in the neighborhood of the origin and is not designated by any number on the graph. This domain is separated from the domain with the number of equilibria equal to 20 by a curve that is very close to an astroid. These are followed by the domains with the number of equilibria equal to 16 and 12. There are only 8 equilibrium positions beyond the boundaries of the domain with the number of equilibria 12. In Fig. 2 ($\nu = 0.2$, $h_3 = 0.4$), the sizes of the domains with the number of equilibria equal to 24, 20, 16, and 12 become smaller than the respective domains in Fig. 1. For the bifurcation value $h_3 = 0.8$, the domain with the number of equilibria 24 vanishes (Fig. 3). In Fig. 3 ($\nu = 0.2$, $h_3 = 0.8$), the domain with the number of equilibria 20 is in the neighborhood of the origin and is not designated by any number on the graph.

There are only five types of domains with the number of equilibria equal to 20, 16, 12, and 8 in the interval $0.8 < h_3 < 1.048$. For the bifurcation value $h_3 = 1.048$, the domain with 20 equilibrium states vanishes (Fig. 4). Figure 4 shows the domains with the number of equilibria equal to 16, 12, and 8. There are three types of domains with the number of equilibria 16, 12, and 8 in the interval $1.048 < h_3 < 3.264$. For the bifurcation value $h_3 = 3.264$, the domain with 16 equilibrium states vanishes (Fig. 5). There are only two types of domains with the number of equilibria equal to 12 and 8 left in the interval $3.264 < h_3 < 4$.

Figures 6–13 show evolution of the domains with the constant number of equilibria for the values of the inertia parameter $\nu = 0.5$ and $\nu = 0.8$. In Figs. 8 and 12, the domain with 16 equilibrium states is located in the neighborhood of the origin and is not designated on the graphs.

When the value of the parameter of gyrostatic moment h_3 exceeds 4, there are only 8 equilibrium orientations for the gyrostat satellite that correspond to four real roots of Eq. (10). This is in agreement with physical considerations—for large values of the vector of gyrostatic moment, equilibria of the satellite are bounded by the direction and value of this moment.

To conclude, we can say, based on the results obtained in this work with the help of symbolic–numerical methods, that the number of equilibrium positions of the gyrostat satellite does not exceed 24 and cannot be smaller than 8 in the general case.

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