Nikolay I. Khokhlov* and Igor B. Petrov

On one class of high-order compact grid-characteristic schemes for linear advection

Abstract: A class of two-point compact difference schemes of the second–third orders of accuracy on a two-point coordinate stencil is considered for a one-dimensional transfer equation. All difference schemes are based on interpolation polynomials constructed on a given stencil. Based on the behaviour of the solution and the character of interpolation polynomials, we propose hybrid compact difference schemes of 2–3rd orders of accuracy on smooth functions producing solutions weakly smoothing the front of discontinuities. The study of grid convergence for the constructed difference scheme is carried out and the propagation of a pulse of complex form is simulated numerically for studying the behaviour of the difference scheme on discontinuous solutions.

Keywords: Compact schemes, grid-characteristic methods, monotone methods, advection equation.

MSC 2010: 35L65

DOI: 10.1515/rnam-2016-0033
Received October 30, 2015; accepted October 11, 2016

Hyperbolic systems of partial differential equations are the base for many mathematical models for solution of various dynamic problems. In order to model physical processes correctly, numerical methods should be stable and have satisfactory accuracy. One of directions in construction of numerical methods of higher orders of accuracy is the use of extended systems of equations [2]. The schemes constructed for such systems use differential corollaries of original equations [3, 6, 7, 11], which allows us to use stencils with the minimal number of nodes, but preserve a higher order of accuracy. Schemes of such type are usually called compact ones [14].

Constructing compact difference schemes for systems of hyperbolic equations, one generally uses three-point [11, 14] and two-point [7, 10] stencils in the coordinate direction. The maximal order of schemes on a two-point stencil presented in literature for systems of hyperbolic equations is four [7, 10]. Integral corollaries from original differential equations were used in [10], that allowed the authors to propose a monotone scheme of the first order in time and a monotonized scheme of the third order in time and both those schemes have the fourth order in the coordinate. In review [7], the fourth order of a scheme was also achieved due to the use of the original system and its differential corollary. The difference scheme for the original system uses the values from the differential corollary on two time layers. The scheme is monotonized and uses a grid-characteristic criterion of monotonicity [16].

In the present paper we consider compact schemes of the second–third orders of accuracy constructed with the use of interpolation polynomials. The solution to the scheme is approximated by a set of interpolation polynomials of different degrees and then, in order to provide monotone behaviour, the scheme is hybridized [4, 9] by choosing one or other polynomial depending on the character of solution. We use the hybridization based on the grid-characteristic criterion [16] and the criterion based on determination of the extremum of an interpolation polynomial on the chosen interval, the latter criterion is proposed in this paper. All schemes are constructed on the minimal two-point coordinate stencil and hence they relate to the class of bi-compact [10] difference schemes. As the result of such construction of the difference scheme, we obtained a monotone behaviour of the difference scheme and a weak ‘blur’ of the front of a discontinuous solution in time.

***Corresponding Author: Nikolay I. Khokhlov:** Moscow Institute of Physics and Technology, Dolgoprudny 141700, Moscow Region, Russia. E-mail: k_h@inbox.ru
Igor B. Petrov: Moscow Institute of Physics and Technology, Dolgoprudny 141700, Moscow Region, Russia

1 Extended system of equations

Let us consider the simplest homogeneous linear transfer equation

$$u_t + \lambda u_x = 0. \tag{1.1}$$

We assume that $\lambda = \text{const} > 0$ because for negative λ all constructions are similar and can be implemented with the change of λ by $-\lambda$ and by using the grid stencil symmetric in x with respect to the point (t^n, x_m).

Along with equation (1.1) we consider its differential corollary. In addition to $u(t, x)$ we introduce the new required function $v(t, x) = u_x(t, x)$ and, differentiating equation (1.1) with respect to x, we obtain for $v(t, x)$ the similar transfer equation

$$v_t + \lambda v_x = 0. \tag{1.2}$$

2 Interpolation polynomials

We consider the extension of difference schemes on a two-point stencil (Fig. 1) with the spatial mesh size h and time step τ:

$$(t^n, x_{m-1}), \quad (t^n, x_m), \quad (t^{n+1}, x_m). \tag{2.1}$$

We use the coordinate system where the point (t^n, x_m) has the coordinates $(0, 0)$ and the point (t^n, x_{m-1}) has the coordinates $(0, -h)$, respectively.

In the interval $(-h, 0)$ we consider interpolation polynomials $f(x)$ approximating the function $u(x)$. Below we omit the index in time where this is possible. The solution to equation (1.1) can be obtained at the time step $n + 1$ as

$$u_m^{n+1} = f(-\lambda\tau)$$

and the solution to the extended system is

$$v_m^{n+1} = f'(-\lambda\tau).$$

The maximal degree of a polynomial that can be constructed on this stencil is three. Represent this polynomial in the form

$$F_3(x) = a_3 x^3 + b_3 x^2 + c_3 x + d_3 \tag{2.2}$$

for original system (1.1); in this case differential approximation (1.2) is approximated by its derivative

$$F_3'(x) = 3a_3 x^2 + 2b_3 x + c_3. \tag{2.3}$$

The coefficients of this polynomial can be obtained from the conditions

$$\begin{aligned} f(-h) &= u_{m-1}^n, \qquad & f(0) &= u_m^n \\ f'(-h) &= v_{m-1}^n, \qquad & f'(0) &= v_m^n. \end{aligned} \tag{2.4}$$

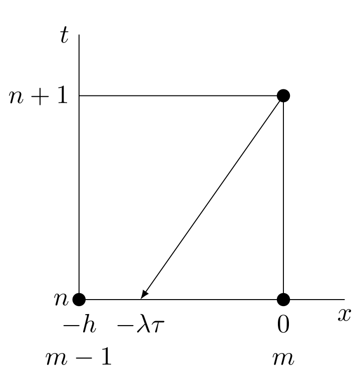

Figure 1. Stencil of the difference scheme.

Based on (2.4), we get

$$
\begin{aligned}
a_3 &= \frac{v_m + v_{m-1}}{h^2} - \frac{2(u_m - u_{m-1})}{h^3} \\
b_3 &= \frac{2v_m + v_{m-1}}{h} - \frac{3(u_m - u_{m-1})}{h^2} \\
c_3 &= v_m \\
d_3 &= u_m.
\end{aligned}
$$

The difference scheme using polynomial (2.2) was apparently first proposed in [17] and described both for linear, and quasilinear systems of hyperbolic equations. In the original paper this scheme was called CIP (Cubic-Interpolated Pseudo-particle); below we also use this name. In further works the CIP scheme was extended to the many-dimensional case and some improvements of this scheme were also proposed [18]. In all mentioned papers the CIP scheme is used without restrictors, or with a restrictor constructed on an extended stencil using also the point (t^n, x_{m+1}) [17]. In this paper we propose a method to obtain a monotone behaviour of the scheme by decreasing the degree of interpolation polynomial (2.2) in the domain of discontinuous solutions and using compact stencil (2.1).

The results of application of the CIP scheme for a pulse of complex form is presented in Fig. 2b.

Along with polynomial (2.2), we consider polynomials of the second order of accuracy and denote them by $F_{2l}(x)$ and $F_{2r}(x)$:

$$F_{2l}(x) = a_{2l}x^2 + b_{2l}x + c_{2l} \tag{2.5}$$

$$F_{2r}(x) = a_{2r}x^2 + b_{2r}x + c_{2r}. \tag{2.6}$$

The polynomial $F_{2l}(x)$ is constructed with the use of the points u_{m-1}, u_m, v_{m-1}, and for $F_{2r}(x)$ we use the points u_{m-1}, u_m, v_m. The coefficients of polynomials (2.5) and (2.6) can be calculated from the conditions

$$
\begin{aligned}
&F_{2l}(0) = u_m, \qquad F_{2l}(-h) = u_{m-1}, \qquad F'_{2l}(-h) = v_{m-1} \\
&F_{2r}(0) = u_m, \qquad F_{2r}(-h) = u_{m-1}, \qquad F'_{2r}(0) = v_m.
\end{aligned}
$$

They have the following form:

$$
\begin{aligned}
a_{2l} &= \frac{u_m - u_{m-1}}{h^2} - \frac{v_{m-1}}{h} \\
b_{2l} &= \frac{2(u_m - u_{m-1})}{h} - v_{m-1} \\
a_{2r} &= \frac{v_m}{h} - \frac{u_m - u_{m-1}}{h^2} \\
b_{2r} &= v_m \\
c_{2l} &= c_{2r} = u_m.
\end{aligned}
$$

The polynomials for the differential corollary can be obtained by differentiation of (2.5) and (2.6), i.e.,

$$F'_{2l}(x) = 2a_{2l}x + b_{2l} \tag{2.7}$$

$$F'_{2r}(x) = 2a_{2r}x + b_{2r}. \tag{2.8}$$

The schemes constructed on these polynomials have the second order of accuracy, but possess a noticeable dispersion. Below we denote the schemes using polynomials (2.5) and (2.6) by CIP2L and CIP2R, respectively.

Along with these polynomials we consider the following first order polynomial on the considered stencil:

$$F_1(x) = a_1x + b_1. \tag{2.9}$$

Based on conditions on the interval boundaries

$$F_1(0) = u_m, \qquad F_1(-h) = u_{m-1}$$

we calculate its coefficients

$$a_1 = \frac{u_m - u_{m-1}}{h}$$
$$b_1 = u_m.$$

The approximation by this polynomial is a first order scheme of CIR (Courant–Isaacson–Rees) type. Differential corollary (1.2) is approximated in this case by the formula

$$F_1'(x) = a_1.$$

Denote this scheme by CIR.

Further, based on these polynomials, we propose approaches to construct compact difference schemes on stencil (2.1) in order to obtain a monotone behaviour of the CIP scheme by choosing different interpolation polynomials depending on the behaviour of solutions of those schemes.

3 Study of the behaviour of interpolation polynomials

Approximate the first derivative of the function $u(t, h)$ in the interval $(-h, 0)$ by the first order formula

$$v_* = a_1 = (u_m - u_{m-1})/h.$$

Below we consider the behaviour of different polynomials depending on required values on the chosen stencil.

The signs of v_*, v_m, and v_{m-1} coincide, that means the fulfillment of the conditions

$$v_m v_{m-1} \geqslant 0, \qquad v_* v_m \geqslant 0. \tag{3.1}$$

In order to increase the order of accuracy, we may use the cubic polynomial $F_3(x)$, however, under the fulfillment of condition (3.1) the function $F_3(x)$ can have extrema in the interval $(-h, 0)$. The condition of existence of an extremum is equivalent to the condition of sign change for the derivative in the considered interval. For example, we may write this condition in the following form (the value at the vertex of the parabola does not coincides in sign to its values on the boundaries of the interval and the vertex of the parabola lies inside the interval):

$$-h \leqslant x_0 \leqslant 0, \qquad F_3'(x_0)v_* < 0 \tag{3.2}$$

where $x_0 = -b_3/(2a_3)$ is the coordinate of the vertex of the parabola.

If condition (3.2) holds true, we have to decrease the degree of the polynomial. Consider the applicability of second degree polynomials. If

$$\min(v_m, v_{m-1}) \leqslant v_* \leqslant \max(v_m, v_{m-1}) \tag{3.3}$$

then polynomials (2.5) and (2.6) lie at different sides from line (2.9) on the whole interval $(-h, 0)$. A convex combination of these polynomials is also a second degree polynomial, i.e.,

$$F_{2lr}(x) = \alpha F_{2l}(x) + (1 - \alpha)F_{2r}(x) \tag{3.4}$$

for all $0 \leqslant \alpha \leqslant 1$. Since the polynomials lie at the different sides from line (2.9), we can take α so that the condition $F_{2lr}(x) = F_1(x)$ holds in the interval $(-h, 0)$ and, using a monotone polynomial of first degree, we can approximate the solution by second order polynomials. This is the case because the coefficient at x^2 vanishes in polynomial (3.4).

If condition (3.3) does not hold, then polynomials (2.5) and (2.6) lie at the same side from line (2.9) in the interval $(-h, 0)$. In this case we may take the polynomial that has no extremum in the interval $(-h, 0)$. If both polynomials have an extremum, we may use first order scheme (2.9). However, as shown by test calculations, at least for linear equations, we may take the polynomial constructed from the derivative lying closer to v_*; this condition can be written as

$$F(x) = \begin{cases} F_{2l}(x), & \Delta_{m-1} \leqslant \Delta_m \\ F_{2r}(x), & \Delta_{m-1} > \Delta_m \end{cases} \tag{3.5}$$

where $\Delta_i = |v_i - v_*|$. The use of such approach provides the second order of accuracy, but does not guarantee the absence of an extremum on the segment.

Let us consider the next case. The signs of v_m and v_{m-1} are different, i.e.,

$$v_m v_{m-1} < 0. \tag{3.6}$$

In this case, using interpolation of order exceeding one, we always have an extremum on the considered segment. However, since the derivatives change their signs, the presence of an extremum is not a non-physical oscillation and, as shown by the test calculations, the use of polynomials of degrees greater than one does not cause non-physical oscillations in the solution. Under the same conditions, the second degree polynomials lie at the same side from the line determined by polynomial (2.9) in the interval $(-h, 0)$; due to this fact, we cannot use their convex combination. However, we can use the third degree polynomial (2.2), or choose the second degree polynomial from condition (3.5). In order to obtain a monotone solution, we have to use the first order interpolation (2.9).

Finally, let the signs of v_m, v_{m-1} be the same, but differ from the sign of v_*, i.e.,

$$v_m v_{m-1} \geqslant 0, \qquad v_* v_m \leqslant 0. \tag{3.7}$$

Since the third degree polynomial (2.2) has two extrema in the interval $(-h, 0)$, it cannot be used. The second degree polynomials (2.5) and (2.6) have extrema in the considered interval too, however, they lie at the different sides from line (2.9). Using their convex combination (3.4), we can always set the coefficient at the quadratic term to zero in the obtained polynomial and then use monotone linear polynomial (2.9), which gives the second order of approximation in this case.

Based on the above analysis and choosing one or other polynomial depending on the character of the solution, we can construct a hybrid scheme producing the solution possessing monotone behaviour.

4 Hybrid schemes

The hybridization considered in this paper consists in the choice of some or other interpolation polynomial depending on the character of the solution on the basis of the fact presented above.

Let us consider two hybrid schemes.

4.1 Grid-characteristic monotonization

Grid-characteristic schemes are based on the use of the characteristic criterion of monotonicity [7, 8], namely,

$$\min(u_m, u_{m-1}) \leqslant u_m^{n+1} \leqslant \max(u_m, u_{m-1}). \tag{4.1}$$

A detailed description of the difference schemes construction based on the use of characteristic criterion on monotonicity (4.1) (including those for extended systems of equations) was presented in [7]. This paper also presented finite-difference schemes of third and fourth orders of accuracy for extended systems of equations (1.1) and (1.2) on compact stencil (2.1). In contrast with [7], in this paper we use an interpolation polynomial for solving system (1.1) and its derivative for solving differential corollary (1.2). In [7], systems (1.1) and (1.2) were solved independently using various difference schemes. In this case the hybridization of schemes for the original system of equations and its differential corollary is performed independently.

The hybridization based on grid-characteristic criterion (4.1) is performed for schemes described in this paper in the following way. We calculate the values u_m^{n+1} with the use of some interpolation polynomial; the fulfillment of the given monotonicity criterion is checked for this value. If the criterion is hot fulfilled, the degree of the polynomial is decreased until it is fulfilled. This operation is performed successively for the polynomials $F_3(x)$, $F_{2l}(x)$, $F_{2r}(x)$, and $F_1(x)$. After we have chosen the polynomial, for interpolation we calculate the value v_m^{n+1} by using its derivative. Below we denote this scheme by BIS1 (bi-compact interpolation scheme).

4.2 Hybrid scheme of the third order of accuracy

In this paper we also use the hybridization of the scheme of third order of accuracy on the basis of the behaviour of polynomials in the considered interval. The scheme is constructed in the following way. If conditions (3.1) and (3.2) hold, we use the scheme of the second order on the basis of (3.5). If (3.1) hold, but (3.2) does not, we use third order scheme (2.2). If condition (3.3) holds, we use first degree polynomial (2.9), but, as shown above, it provides the second order. If condition (3.6) holds, we use second degree polynomials on the basis of (3.5). Finally, if condition (3.7) holds, we use first order polynomial (2.9) also providing the second order. Denote this scheme by BIS2. The scheme was tested on initial data representing pulse of complex form (5.1) for different values of the Courant number. In all these cases the scheme demonstrated a monotone behaviour.

5 Testing of schemes

We tested the qualitative behaviour of difference schemes for the following initial data representing a pulse of complex form [1, 5, 12]:

$$u(0,x) = \begin{cases} \exp(-\ln 2(x+0.7)^2/0.0009), & -0.8 \leqslant x \leqslant -0.6 \\ 1, & -0.4 \leqslant x \leqslant -0.2 \\ 1-|10x-1|, & 0.0 \leqslant x \leqslant 0.2 \\ (1-100(x-0.5)^2)^{1/2}, & 0.4 \leqslant x \leqslant 0.6 \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

This initial condition consists of a discontinuous rectangular pulse, a triangular pulse, a pulse of Gaussian distribution form, and a half-elliptic pulse. We used the grid consisting of 200 nodes with periodic boundary conditions. The solution is presented for the time moment $t = 4.0$, the time step was taken relative to the Courant number 0.4. The rate of transfer was $\lambda = 1$ in all tests. The results for all schemes are presented in Fig. 2.

As seen from the graphs in Fig. 2, the schemes constructed on polynomials of degree exceeding one possess the property of dispersion. In this case the third order scheme has the least oscillations and in some cases can be applied without monotonization. Hybrid schemes have monotone behaviour, namely, BIS1 is monotone with respect to the grid-characteristic criterion, BIS2 also demonstrates monotone behaviour and for short pulses its behaviour is better than that of BIS1. The schemes demonstrate similar results for other Courant numbers, there are no oscillations for BIS1 and BIS2, as well.

Norms of errors are presented for this test in Table 1. We used the following norms: $L_1 = \sum_i |x_i|$, $L_2 = (\sum_i x_i^2)^{1/2}$, $L_\infty = \max |x_i|$, $x_i = u_i - u_i^{\text{theor}}$, u_i is the solution obtained numerically, u_i^{theor} is the exact solution, i is the grid node number, the summation is taken over all nodes.

As seen from calculations, the CIP scheme provides the least norms of errors. However, the calculations performed by BIS2 demonstrate the least errors among non-oscillating schemes.

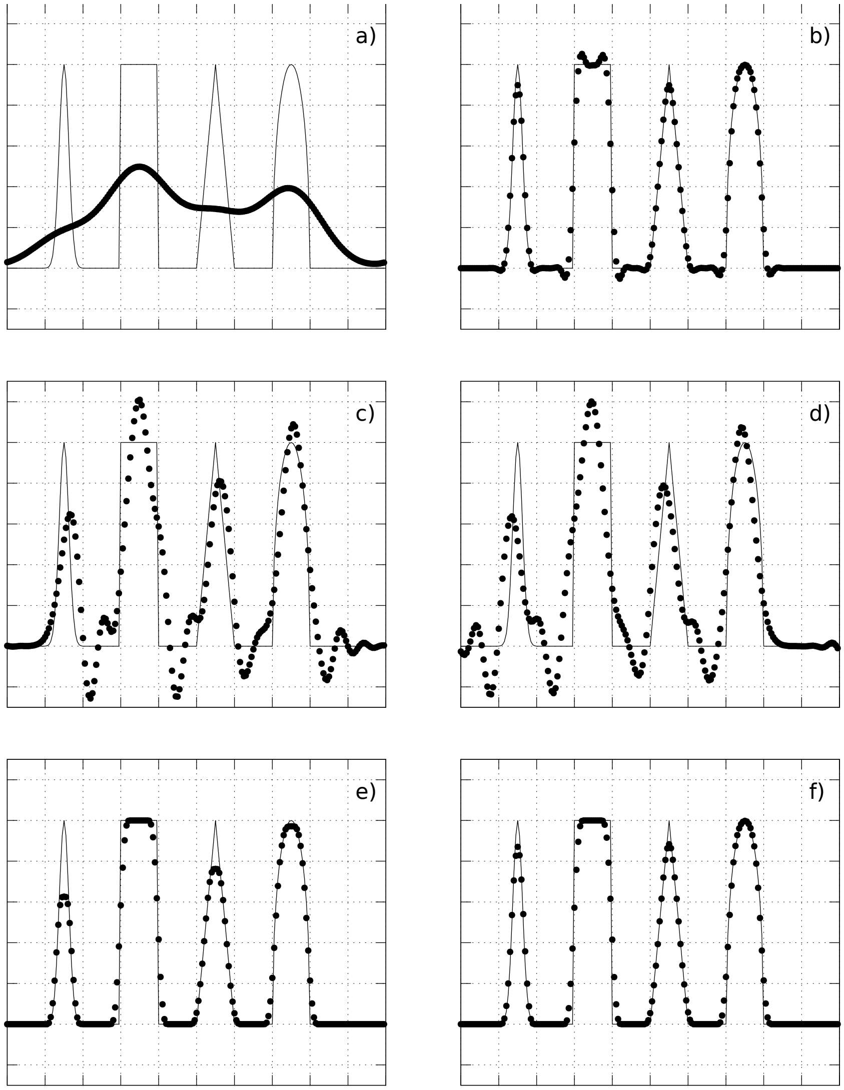

Figure 2. Numerical (dots) and exact (solid line) solutions to linear transfer equation (1.1) for initial data (5.1) after 1000 steps of the difference scheme. a) CIR; b) CIP; c) CIP2L; d) CIP2R; e) BIS1; f) BIS2.

	L_1	L_2	L_∞
CIR	0.562	0.092	0.810
CIP	0.055	0.019	0.389
CIP2L	0.262	0.050	0.635
CIP2L	0.271	0.051	0.644
BIS1	0.068	0.024	0.417
BIS2	0.054	0.020	0.429

Table 1. Norms of errors for different schemes.

5.1 Pulse of right triangular form

In addition, we carried out the test on pulse of right triangular form (5.2):

$$u(0,x) = \begin{cases} (x+0.4)/0.8, & -0.4 \leqslant x \leqslant 0.4 \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

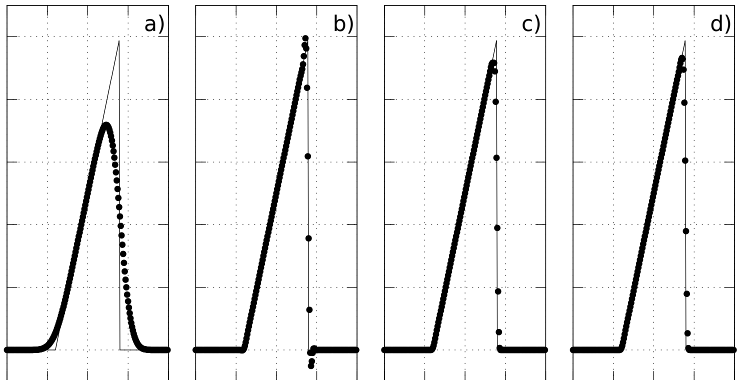

Figure 3. Numerical (dots) and exact (solid line) solutions to linear transfer equation (1.1) for initial data (5.2) after 1000 steps of the difference scheme. a) CIR; b) CIP; c) BIS1; d) BIS2.

The other parameters were similar to those of previous test. The results of CIR, CIP schemes and for BIS1, BIS2 are presented in Fig. 3.

The CIP scheme gives small oscillations, but better describes the position of the top point of the triangle. The behaviours of the BIS1 and BIS2 schemes are similar, they produce no oscillations, but the BIS1 scheme ‘cuts’ the top of the triangle a few more.

5.2 Grid convergence on a uniform grid

Table 2 presents the results of the grid convergence testing for the considered schemes.

The initial conditions were specified by the pulse

$$\begin{aligned} u(0, x) &= \sin^4(\pi x) \\ v(0, x) &= 4\pi \sin^3(\pi x)\cos(\pi x) \end{aligned} \tag{5.3}$$

in the domain $[-1, 1]$ and by the periodic boundary conditions. The Courant number was 0.2, the calculations were performed until the time moment $t = 2.0$. As seen from the results of testing, the numerical order of convergence of the BIS2 scheme is greater than for BIS1 and corresponds to the order of the CIP scheme also in the norm L_∞. The order of convergence was calculated by the formula $p = (\log(E_h) - \log(E_{h/2}))/\log(1/2)$, where E_h is the norm of error for a grid with the step h. In this case, BIS2 does not demonstrate oscillations typical for the CIP scheme.

5.3 Grid convergence on a grid with irregular steps

In the previous section we have shown that BIS2 demonstrates the third order of convergence on a uniform grid. In this section we show that the order of this scheme is the same for non-uniform grids. We use grids similar to those of original paper [18] considering the CIP scheme. The parameters of grids are presented below. In the case of translation of the ith node we assume $h = \Delta x_i$ in calculations formulas.

In the first case the step of the grid changed abruptly in the passage from one grid node to another. The step was taken from the relation $\Delta x_i = hr(i)$, where $h = 2.0/\sum_{i=0}^{N} r(i)$,

$$r(i) = \begin{cases} 1.0, & IL \leqslant i \leqslant IR \\ \alpha, & \text{otherwise.} \end{cases}$$

The coefficient α and the number of the grid nodes N varied as $\alpha = \{0.5, 1.0, 1.01, 1.05, 1.2, 1.5\}$ and $N = \{100, 200, 400, 800, 1600\}$, respectively. The grid parameters were $IL = N/4$, $IR = IL + 20NM - 1$, $NM =$

	N	L_1	order L_1	L_∞	order L_∞
CIR	100	3.28e-01	—	3.49e-01	—
	200	2.00e-01	0.71	2.25e-01	0.63
	400	1.13e-01	0.82	1.32e-01	0.77
	800	6.06e-02	0.90	7.18e-02	0.87
	1600	3.14e-02	0.95	3.76e-02	0.93
CIP	100	5.98e-04	—	5.78e-04	—
	200	7.51e-05	2.99	7.26e-05	2.99
	400	9.40e-06	3.00	9.09e-06	3.00
	800	1.18e-06	3.00	1.14e-06	3.00
	1600	1.47e-07	3.00	1.42e-07	3.00
CIP2L	100	2.13e-02	—	2.17e-02	—
	200	5.37e-03	1.99	5.43e-03	2.00
	400	1.34e-03	2.00	1.36e-03	2.00
	800	3.36e-04	2.00	3.39e-04	2.00
	1600	8.39e-05	2.00	8.48e-05	2.00
CIP2R	100	3.10e-02	—	3.17e-02	—
	200	7.98e-03	1.96	8.09e-03	1.97
	400	2.01e-03	1.99	2.03e-03	1.99
	800	5.03e-04	2.00	5.09e-04	2.00
	1600	1.26e-04	2.00	1.27e-04	2.00
BIS1	100	3.42e-03	—	2.37e-02	—
	200	6.67e-04	2.36	7.97e-03	1.57
	400	1.39e-04	2.27	2.91e-03	1.45
	800	2.92e-05	2.25	1.03e-03	1.50
	1600	5.92e-06	2.30	3.55e-04	1.54
BIS2	100	5.72e-04	—	8.47e-04	—
	200	7.24e-05	2.98	9.81e-05	3.11
	400	9.17e-06	2.98	1.16e-05	3.08
	800	1.16e-06	2.99	1.39e-06	3.06
	1600	1.46e-07	2.99	1.68e-07	3.05

Table 2. Grid convergence.

$N/100$. The other parameters such as the time step, the number of steps and the form of the pulse were similar to the previous test.

The results of testing for BIS2 are presented in Table 3. As seen from calculations, the scheme retains the order for all values of the parameter α.

In the second case the grid step varied smoothly according to the law

$$r(i) = \begin{cases} 1.0 + \beta \sin(2\pi(i - IL)/(IR - IL)), & IL \leqslant i \leqslant IR \\ 1.0, & \text{otherwise} \end{cases}$$

where $\beta = \{0.0, 0.05, 0.2, 0.35, 0.5\}$. The tests were performed for the same sizes of grids as in the previous case. The grid parameters were $IL = N/4$, $IR = IL + 60NM - 1$, $NM = N/100$.

The results of the BIS2 scheme testing are presented in Table 4. The scheme retains the order of the scheme for all values of β.

5.4 Grid convergence in the case of discontinuous solution

In addition, we carried out tests of grid convergence in the case of discontinuous solution to the transfer equation. We specified pulse of complex form (5.1) in the domain $[-1, 1]$. At points of discontinuity the initial condition for the derivative was specified by zero. Table 5 presents the results of testing for calculation times $t = 20$ (10 periods) and $t = 2000$ (1000 periods).

The test used the integral norm L_1. In the case of discontinuous solution, all schemes must show the first order of convergence in this norm [13, 15]. The Courant number was 0.4 in these calculations. All schemes,

α	N	L_1	order L_1	L_∞	order L_∞
0.50	100	7.29e-04	—	1.04e-03	—
	200	9.28e-05	2.97	1.27e-04	3.03
	400	1.17e-05	2.98	1.51e-05	3.07
	800	1.48e-06	2.99	1.82e-06	3.06
	1600	1.87e-07	2.99	2.19e-07	3.05
1.00	100	5.72e-04	—	8.47e-04	—
	200	7.24e-05	2.98	9.81e-05	3.11
	400	9.17e-06	2.98	1.16e-05	3.08
	800	1.16e-06	2.99	1.39e-06	3.06
	1600	1.46e-07	2.99	1.68e-07	3.05
1.01	100	5.70e-04	—	8.28e-04	—
	200	7.25e-05	2.97	9.90e-05	3.06
	400	9.18e-06	2.98	1.16e-05	3.09
	800	1.16e-06	2.99	1.39e-06	3.06
	1600	1.46e-07	2.99	1.69e-07	3.04
1.05	100	5.71e-04	—	8.23e-04	—
	200	7.26e-05	2.97	9.66e-05	3.09
	400	9.20e-06	2.98	1.17e-05	3.04
	800	1.16e-06	2.99	1.41e-06	3.06
	1600	1.46e-07	2.99	1.70e-07	3.05
1.20	100	5.96e-04	—	8.71e-04	—
	200	7.58e-05	2.97	1.03e-04	3.08
	400	9.60e-06	2.98	1.23e-05	3.07
	800	1.21e-06	2.99	1.47e-06	3.06
	1600	1.52e-07	2.99	1.78e-07	3.05
1.50	100	7.39e-04	—	1.09e-03	—
	200	9.37e-05	2.98	1.31e-04	3.05
	400	1.19e-05	2.98	1.56e-05	3.08
	800	1.50e-06	2.99	1.86e-06	3.06
	1600	1.88e-07	2.99	2.24e-07	3.05

Table 3. Grid convergence on a nonuniform grid for BIS2 in the case of sharp change of the grid step size.

i.e., CIP, BIS1, and BIS2 presented in the table have orders of convergence close to one when the grid is refined. The BIS1 scheme has somewhat worse convergence than in other schemes. The CIP and BIS2 schemes have approximately the same norms of errors and orders of convergence.

A similar test was carried out for a pulse of rectangular form, i.e.,

$$u(0,x) = \begin{cases} 1, & -0.9 \leqslant x \leqslant -0.8 \\ 0, & \text{otherwise} \end{cases}$$

the other parameters were not changed. The results are presented in Table 6.

The tests show a lower order than in the case of a pulse of complex form (5.1), but it is comparable for all schemes considered in the tests.

5.5 Numerical study of the conservative property of the obtained schemes

The BIS1 and BIS2 schemes are not conservative. In this paper we study the conservative property of schemes numerically on the example of transfer of the rectangular pulse

$$u(0,x) = \begin{cases} 1, & -0.1 \leqslant x \leqslant 0.1 \\ 0, & \text{otherwise.} \end{cases}$$

The other parameters are similar to those from the previous test on a uniform grid. Figure 4 presents the portion of the area below the pulse in percents relative to its initial value depending on the distance of transfer and normed by the size of the initial profile.

β	N	L_1	order L_1	L_∞	order L_∞
0.00	100	5.72e-04	—	8.47e-04	—
	200	7.24e-05	2.98	9.81e-05	3.11
	400	9.17e-06	2.98	1.16e-05	3.08
	800	1.16e-06	2.99	1.39e-06	3.06
	1600	1.46e-07	2.99	1.68e-07	3.05
0.05	100	5.74e-04	—	8.46e-04	—
	200	7.28e-05	2.98	9.85e-05	3.10
	400	9.23e-06	2.98	1.18e-05	3.07
	800	1.16e-06	2.99	1.41e-06	3.06
	1600	1.46e-07	2.99	1.70e-07	3.05
0.20	100	6.22e-04	—	9.40e-04	—
	200	7.88e-05	2.98	1.07e-04	3.13
	400	9.97e-06	2.98	1.28e-05	3.06
	800	1.26e-06	2.99	1.53e-06	3.07
	1600	1.58e-07	2.99	1.86e-07	3.04
0.35	100	7.27e-04	—	1.10e-03	—
	200	9.21e-05	2.98	1.28e-04	3.11
	400	1.17e-05	2.98	1.55e-05	3.04
	800	1.47e-06	2.99	1.84e-06	3.08
	1600	1.85e-07	2.99	2.21e-07	3.06
0.50	100	8.93e-04	—	1.36e-03	—
	200	1.13e-04	2.98	1.61e-04	3.08
	400	1.43e-05	2.98	1.92e-05	3.07
	800	1.81e-06	2.99	2.28e-06	3.08
	1600	2.28e-07	2.99	2.77e-07	3.04

Table 4. Grid convergence on a nonuniform grid for BIS2 in the case of smooth variation of the grid step.

The CIP scheme is conservative [18], the BIS1 and BIS2 schemes are not. However, the area of the initial profile changes insignificantly (by less than 2%).

Figure 5 presents the graph of solution after 250000 steps of the difference scheme (transfer by the distance equal to 5000 sizes of the initial profile).

5.6 Numerical study of the scheme monotonicity

In addition we carried out the numerical study of the monotonicity of schemes considered here. The initial condition was specified in the form of parabola

$$\begin{aligned} u(0,x) &= 4(x-h/2)^2/h - h \\ v(0,x) &= 8(x-h/2)/h \end{aligned} \tag{5.4}$$

where h is the grid step. The grid was chosen so that the point 0 was at a grid node. For such initial condition all values specified on the grid are non-negative. After that we considered the translation of the solution by the distance of the half-step $h/2$ of the grid near the vertex of the parabola. The result is presented in Fig. 6.

The second and higher order schemes reconstruct the vertex of the parabola after its translation by half-step. This vertex was previously at the middle of a cell, but now it is translated to a grid node. In this case the solution at the vertex of the parabola is negative. Such behaviour of the solution completely corresponds to Godunov's theorem and indicates the non-monotonicity of the scheme. The monotone schemes do not produce negative values, but the order of approximation decreases.

The graph of solution presented in Fig. 6 shows that only the CIR and BIS schemes give a monotone solution. BIS1 decreases the approximation order only at the vertex of the parabola, in other nodes of the scheme the solution is translated exactly in contrast with the CIR scheme. The CIP and BIS2 schemes translate the parabola exactly and are not monotone.

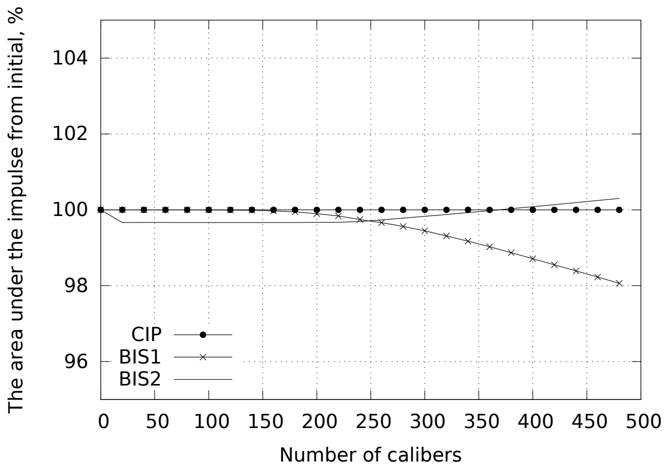

Figure 4. The portion of the area under the pulse graph relative to the initial one in percents depending on the distance of transfer normed by the size of the initial profile. Results for the BIS1, BIS2, and CIP schemes.

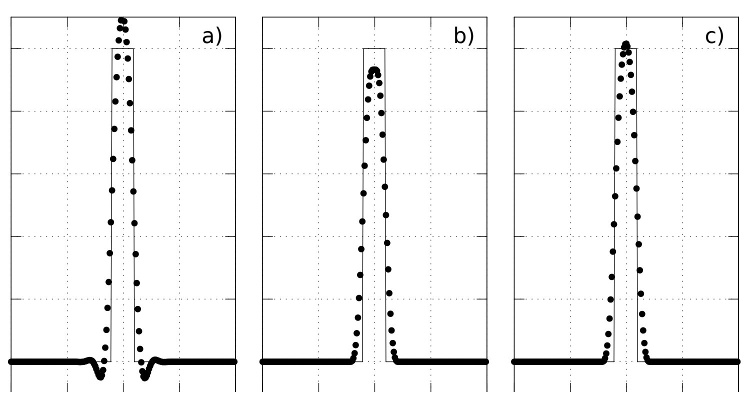

Figure 5. Numerical (dots) and exact (solid line) solutions to the linear transfer equation (1.1) after 250000 steps of the difference scheme. a) CIP; b) BIS1; c) BIS2.

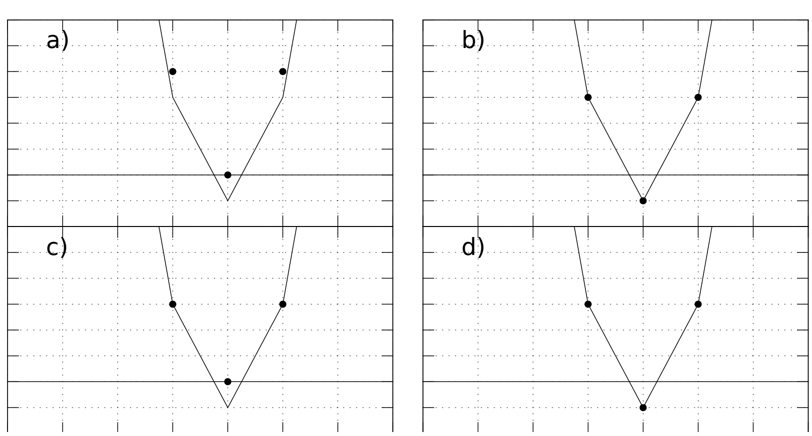

Figure 6. Numerical (dots) and exact (line) solutions to linear equation (1.1) after translation of the solution by half-cell step. The horizontal line indicates the zero level. a) CIR; b) CIP; c) BIS1; d) BIS2.

Table 5. Grid convergence for a pulse of complex form.

Scheme	N	$t = 20$		$t = 2000$	
		L_1	order L_1	L_1	order L_1
CIP	100	1.93e-01	—	5.75e-01	—
	200	9.48e-02	1.02	3.64e-01	0.66
	400	4.60e-02	1.04	2.08e-01	0.81
	800	2.27e-02	1.02	1.09e-01	0.93
	1600	1.22e-02	0.90	5.31e-02	1.04
BIS1	100	2.90e-01	—	5.78e-01	—
	200	1.21e-01	1.26	5.25e-01	0.14
	400	5.63e-02	1.11	3.08e-01	0.77
	800	2.56e-02	1.14	1.35e-01	1.20
	1600	1.27e-02	1.01	6.24e-02	1.11
BIS2	100	1.98e-01	—	6.06e-01	—
	200	9.35e-02	1.08	4.22e-01	0.52
	400	4.46e-02	1.07	2.14e-01	0.98
	800	2.20e-02	1.02	1.07e-01	1.00
	1600	1.18e-02	0.90	5.10e-02	1.07

Table 6. Grid convergence in the case of rectangular pulse.

Scheme	N	$t = 20$		$t = 2000$	
		L_1	order L_1	L_1	order L_1
CIP	100	7.23e-02	—	1.66e-01	—
	200	3.88e-02	0.90	1.31e-01	0.34
	400	2.68e-02	0.53	8.38e-02	0.65
	800	1.61e-02	0.74	4.28e-02	0.97
	1600	9.56e-03	0.75	3.03e-02	0.50
BIS1	100	9.86e-02	—	1.27e-01	—
	200	5.66e-02	0.80	1.23e-01	0.04
	400	2.65e-02	1.10	1.04e-01	0.25
	800	1.57e-02	0.75	6.23e-02	0.74
	1600	9.36e-03	0.75	2.98e-02	1.06
BIS2	100	7.69e-02	—	1.69e-01	—
	200	4.16e-02	0.89	1.33e-01	0.35
	400	2.64e-02	0.66	8.52e-02	0.64
	800	1.57e-02	0.75	4.62e-02	0.88
	1600	9.36e-03	0.75	2.97e-02	0.64

6 Conclusion

In this paper we consider several compact difference schemes for the one-dimensional transfer equation on a two-point stencil. The difference schemes are constructed with the use of interpolation polynomials of the first–third orders of accuracy. Starting from different difference schemes constructed on the same stencil, we construct hybrid difference schemes possessing the property of monotonicity. We have constructed the hybrid monotone difference scheme BIS1 based on the grid-characteristic monotonicity criterion and the hybrid difference scheme BIS2 based on the hybridization criterion proposed in this paper. The hybridization criterion is based on the determination of local extrema in the interval including the grid stencil, the obtained scheme is not monotone, but it possesses a monotone behaviour and less numerical diffusion. The theoretical order of approximation of the constructed schemes is 1–3 depending on the behaviour of the solution. Numerical study of the convergence of these difference schemes indicated an increased order of accuracy. Thus, the BIS2 gives the order 3 in the norms L_1 and L_∞, the BIS2 gives the order 2.2 in the norm L_1 and 1.5 in the norm L_∞. The BIS1 satisfies monotonicity criterion (4.1), which is sufficient for the stability [7]. We did not study the stability of BIS2 separately, but it proved to be stable in numerical experiments for different Courant numbers less than or equal to one and different forms of pulses.

The original texts of all test examples are available in Internet at https://github.com/khokhlov/compact3.

Acknowledgment: The authors are grateful to Academician of the RAS O. M. Belotserkovskii for attention to this work.

Funding: The work was supported by the Grant of the President of the Russian Federation (MK-3383.2014.9).

References

[1] D. S. Balsara and C.-W. Shu, Monotonicity preserving weighted essentially non-oscillatory schemes with increasingly high order of accuracy. *J. Comput. Phys.* **160** (2000), No. 2, 405–452.
[2] R. Courant and P. Lax, On nonlinear partial differential equations with two independent variables. *Commun. Pure Appl. Math.* **2** (1949), No. 2–3, 255–273.
[3] V. G. Grudnitskii and Yu. A. Prokhorchuk, A method for constructing difference schemes of any order of accuracy for partial differential equations. *Dokl. Akad. Nauk SSSR* **234** (1977), No. 6, 1249–1252 (in Russian).
[4] V. D. Ivanov, V. I. Kondaurov, I. B. Petrov, and A. S. Kholodov, Calculation of dynamic deformation and distructure of elastic–plastic body by grid-characteristic methods. *Matem. Model.* **2** (1990), No. 11, 10–29 (in Russian).
[5] G. S. Jiang and C. W. Shu, Efficient implementation of weighted ENO schemes. *J. Comput. Phys.* **228** (1995), No. 126, 202–228.
[6] A. S. Kholodov, Ya. A. Kholodov, E. L. Stupitskii, and A. Yu. Repin, Numerical modelling of the behaviour of high-energy plasmoid in the upper ionosphere. *Matem. Model.* **16** (2004), No. 8, 3–23,
[7] A. S. Kholodov and Ya. A. Kholodov, Monotonicity criteria for difference schemes designed for hyperbolic equations. *Comput. Math. Math. Phys.* **46** (2006), No. 9, 1560–1588.
[8] K. M. Magomedov and A. S. Kholodov, On the construction of difference schemes for hyperbolic equations based on characteristic relations. *Zh. Vychisl. Mat. Mat. Fiz.* **9** (1969), No. 2, 373–386 (in Russian).
[9] I. B. Petrov and A. S. Kholodov, On the regularization of discontinuous numerical solutions to hyperbolic equations. *Zh. Vychisl. Mat. Mat. Fiz.* **24** (1984), No. 8, 1172–1188 (in Russian).
[10] B. V. Rogov and M. N. Mikhailovskaya, Monotonic bi-compact schemes for linear transport equations. *Math. Models Comput. Simul.* **4** (2012), No. 1, 92–100.
[11] Yu. I. Shokin and N. N. Yanenko, *Method of Differential Approximation: Application to Gas Dynamics*. Nauka, Novosibirsk, 1985 (in Russian).
[12] A. Suresh and H. T. Huynh, Accurate monotonicity-preserving schemes with Runge–Kutta time stepping. *J. Comput. Phys.* **136** (1997), No. 1, 83–99.
[13] V. A. Titarev and E. F. Toro, WENO schemes based on upwind and centred TVD fluxes. *Computers & Fluids* **34** (2005), No. 6, 705–720.
[14] A. I. Tolstykh, *Compact Difference Schemes and Their Application to Aerohydrodynamics Problems*. Nauka, Moscow, 1990 (in Russian).
[15] E. F. Toro and V. A. Titarev, TVD fluxes for the high-order ADER scheme. *J. Sci. Computing* **24** (2005), No. 3, 285–309.
[16] B. Van Leer, Towards the ultimate conservative difference scheme, V. A second-order sequel to Godunov's method. *J. Comput. Phys.* **32** (1979), No. 1, 101–136.
[17] T. Yabe, T. Aoki, G. Sakaguchi, P. Y. Wang, and T. Ishikawa, The compact CIP (Cubic-Interpolated Pseudo-particle) method as a general hyperbolic solver. *Computers & Fluids* **19** (1991), No. 3–4, 421–431.
[18] T. Yabe, H. Mizoe, K. Takizawa, H. Moriki, H.-N. Im, and Yo. Ogata, Higher-order schemes with CIP method and adaptive Soroban grid towards mesh-free scheme. *J. Comput. Phys.* **194** (2004), No. 1, 57–77.